

ON THE COHOMOLOGY GROUPS OF LOCAL SYSTEMS OVER HILBERT MODULAR VARIETIES VIA HIGGS BUNDLES

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ABSTRACT. Let X be a Hilbert modular variety and \mathbb{V} a non-trivial local system over X . In this paper we study Deligne-Saito's mixed Hodge structure (MHS) on the cohomology groups $H^k(X, \mathbb{V})$ using the method of Higgs bundles. Among other results we give a dimension formula for the Hodge numbers and show that the mixed Hodge structure is split over \mathbb{R} . These results are analogous to [18] in the cocompact case and complement the results in [8] for constant coefficients.

1. INTRODUCTION

Consider the Lie group $G = SL(2, \mathbb{R})^n \times U$, where U is connected and compact. In their classical work [18] Matsushima and Shimura study the cohomology groups $H^*(X, \mathbb{V})$ with values in a local system \mathbb{V} attached to a linear representation of G on a compact quotient X of a product of upper half planes by a discrete subgroup $\Gamma \subset G$, see §1 in [18]. The main result of [18] is a dimension formula for the Hodge numbers of the pure Hodge structure on $H^*(X, \mathbb{V})$.

The arguments and results of the present paper grew out of an attempt to find a generalization of the results in [18]. The use of the maximum principle in the proof of the vanishing result of Theorem 3.1 in [18] presents an obvious difficulty for a direct generalization to the non-compact case. The case of Hilbert modular surfaces was already studied in the thesis [28] of the second named author. The final approach we have taken is a mixture of the one of Zucker in [32] for general locally symmetric varieties and the original theory of harmonic forms in [18] for discrete quotients of products of upper half planes. The technique of Higgs bundles made it possible to combine both methods effectively. The vanishing theorem of Mok on locally homogenous vector bundles [19] in the case of Hilbert modular varieties is indispensable to obtain our results for *all* non-trivial local systems, i.e., also non-regular ones. We are aware of the fact that some of our results can be also explained in an automorphic setting. Harris and Zucker [11, 12, 13] have developed a general framework using automorphic forms which is related to our work through the BGG-complex. However our approach is purely Hodge theoretic and can be applied to other Shimura varieties and to more general non-locally-homogenous situations.

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Now let X be a Hilbert modular variety and \mathbb{V} an irreducible local system over X . Then the cohomology group $H^k(X, \mathbb{V})$ carries a natural real mixed Hodge structure (see §2), which is the principal object of study in the present paper. The case of constant local system has been treated in the book [8] by Freitag (see Ch. III in loc. cit.). Thus we shall assume here that \mathbb{V} is a non-trivial local system. Any irreducible complex local system \mathbb{V} over X induced by a linear representation of G is of the form \mathbb{V}_m for certain $m = (m_1, \dots, m_n) \in \mathbb{N}_0^n$, see §2. Our results are summarized as follows (the notations in the statements are collected at the end of the section):

Theorem 1.1. *Let X be a Hilbert modular variety X of dimension n and \mathbb{V}_m be an irreducible non-trivial local system determined by $m = (m_1, \dots, m_n) \in \mathbb{N}_0^n$. Then:*

- (i) $H^k(X, \mathbb{V}_m) = 0$ for $0 \leq k \leq n-1$ and $k = 2n$.
- (ii) If $m_1 = \dots = m_n$, then for $n+1 \leq k \leq 2n-1$

$$h_k^{|m|+n, |m|+n} := \dim_{\mathbb{C}} F^{|m|+n} W_{2(|m|+n)} H^k(X, \mathbb{V}_m) = \dim_{\mathbb{C}} H^k(X, \mathbb{V}_m) = \binom{n-1}{k-n} h,$$

where h is the number of cusps.

- (iii) If not all m_i are equal, then $H^k(X, \mathbb{V}_m) = 0$ for $n+1 \leq k \leq 2n-1$.
- (iv) Furthermore, if $m_1 = \dots = m_n$, then

$$\dim H^n(X, \mathbb{V}_m) = [2(m_1+1)]^n [h^{n,0}(\bar{X}) + (-1)^n] + h.$$

Moreover, for each $P = (m_1+1)l_P$, $0 \leq l_P \leq n$, and $P+Q = \dim H^k(X, \mathbb{V}_m) = |m|+n$,

$$h_n^{P,Q} = \binom{n}{l_P} (m_1+1)^n [h^{n,0}(\bar{X}) + (-1)^n],$$

$$h_n^{|m|+n, |m|+n} = h \text{ and otherwise } h_n^{P,Q} = 0.$$

- (v) If $m_1 = \dots = m_n$ is not satisfied, then

$$\dim H^n(X, \mathbb{V}_m) = 2^n [h^{n,0}(\bar{X}) + (-1)^n] \prod_{i=1}^n (m_i+1).$$

Furthermore, for $P+Q = |m|+n$,

$$h_n^{P,Q} = N(m, P) [h^{n,0}(\bar{X}) + (-1)^n] \prod_{i=1}^n (m_i+1),$$

where $N(m, P)$ is the cardinality of the set $\{I \subset \{1, \dots, n\} \mid |m_I| + |I| = P\}$, and otherwise $h_n^{P,Q} = 0$, if $P+Q \neq |m|+n$.

Each irreducible complex local system $\mathbb{V} = \mathbb{V}_m$ underlies a natural real polarized variation of Hodge structures (\mathbb{R} -PVHS) $\mathbb{V}_{\mathbb{R}}$ of weight $|m|$ (see §2). After Deligne-Saito [5, 21, 22, 23, 24, 31], the cohomology group $H^k(X, \mathbb{V}_{\mathbb{R}})$ carries a natural real MHS with weights $\geq |m| + k$. This

MHS is defined over \mathbb{Q} if all m_i are equal. The natural inclusion $j : X \rightarrow X^*$ into the *Baily-Borel* compactification induces an injective morphism of MHS $IH^k(X^*, \mathbb{V}_{\mathbb{R}}) \rightarrow H^k(X, \mathbb{V}_{\mathbb{R}})$ by Proposition 6.1. We denote again by $IH^k(X^*, \mathbb{V}_{\mathbb{R}})$ the image of the embedding. The theory of Eisenstein cohomology (see [10], [26]) provides a decomposition $H^k(X, \mathbb{V}_{\mathbb{R}}) = H_{\dagger}^k(X, \mathbb{V}_{\mathbb{R}}) \oplus H_{\text{Eis}}^k(X, \mathbb{V}_{\mathbb{R}})$, where $H_{\dagger}^k(X, \mathbb{V}_{\mathbb{R}})$ is the image of $H_c^k(X, \mathbb{V}_{\mathbb{R}})$, the cohomology of $\mathbb{V}_{\mathbb{R}}$ with compact supports, in $H^k(X, \mathbb{V}_{\mathbb{R}})$.

Theorem 1.2. *Let \mathbb{V}_m be an irreducible non-trivial complex local system over X . Let $n \leq k \leq 2n - 1$ and let $(H^k(X, \mathbb{V}_{\mathbb{R}}), W, F^{\cdot})$ be Deligne-Saito's MHS. Then:*

- (i) *For $n + 1 \leq k \leq 2n - 1$, one has $H^k(X, \mathbb{V}_{\mathbb{R}}) = H_{\text{Eis}}^k(X, \mathbb{V}_{\mathbb{R}})$ and the MHS on $H^k(X, \mathbb{V}_{\mathbb{R}})$ is pure Hodge-Tate of type $(|m| + n, |m| + n)$.*
- (ii) *$IH^n(X^*, \mathbb{V}_{\mathbb{R}}) = H_{\dagger}^n(X, \mathbb{V}_{\mathbb{R}})$ and $H^n(X, \mathbb{V}_{\mathbb{R}}) = IH^n(X^*, \mathbb{V}_{\mathbb{R}}) \oplus H_{\text{Eis}}^n(X, \mathbb{V}_{\mathbb{R}})$ is a natural splitting of MHS over \mathbb{R} into two pieces with weights $|m| + n$ and $2(|m| + n)$.*

Due to the splitting of the MHS $(H^k(X, \mathbb{V}_{\mathbb{R}}), W, F^{\cdot})$ over \mathbb{R} , one has a bigrading $H^k(X, \mathbb{V}_m) = \bigoplus_{P+Q=k} H_k^{P,Q}$ with $H_k^{P,Q} = F^P \cap \bar{F}^Q \cap W_{P+Q, \mathbb{C}}$ satisfying

$$W_{l, \mathbb{C}} = \bigoplus_{P+Q \leq l} H_k^{P,Q}, \quad F^P = \bigoplus_{r \geq P} H_k^{r,s}.$$

In addition we can give an algebraic description of the (non-zero) Hodge (P, Q) -components $H_k^{P,Q}$. This result can be regarded as a kind of generalized *Eichler-Shimura isomorphism* for Hilbert modular varieties:

Theorem 1.3. *One has the following natural isomorphisms:*

- (i) *For $n + 1 \leq k \leq 2n - 1$, $H^k(X, \mathbb{V}_m) = H_k^{|m|+n, |m|+n} \simeq H^{k-n}(\bar{X}, \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2})$.*
- (ii) *$H_n^{|m|+n, 0} \simeq H^0(\bar{X}, \mathcal{O}_{\bar{X}}(-S) \otimes \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2})$, $H_n^{|m|+n, |m|+n} \simeq H^0(S, \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2}|_S)$,*
and for $0 \leq P \leq |m| + n - 1$, $P + Q = |m| + n$,

$$H_n^{P,Q} \simeq \bigoplus_{\substack{I \subset \{1, \dots, n\}, \\ |m_I| + |I| = P}} H^{k-|I|}(\bar{X}, \bigotimes_{i \in I} \mathcal{L}_i^{m_i+2} \otimes \bigotimes_{i \in I^c} \mathcal{L}_i^{-m_i}).$$

In the above results, Theorem 1.1, (i) and the first half of Theorem 1.2, (i) for regular local systems are special cases of Li-Schwermer [16] (see also Saper [25]). Wildeshaus has recently informed us that the main result in [2] also implies Theorem 1.1, (ii). Combined with Lemma 6.2, one is able to show then the second half of Theorem 1.2, (i) for regular local systems. The paper is organized as follows: Section 2 contains the basic set-up. In section 3 we compute the logarithmic Higgs cohomology and present an algebraic description of the gradings of the Hodge filtration on cohomology group. At this point one then has certain a priori information of the sheaf cohomologies from known vanishing results. The full information is not known

until one has analyzed the MHS of the cohomology in some detail. Section 4 deals with the intersection cohomology which in turn relies on the theory of L^2 -harmonic forms. This provides an L^2 -generalization of the results in [18]. Section 5 introduces Eisenstein cohomology. This has been extensively investigated in [9], [8] for constant coefficients and in [33, 11, 12, 13], [16] for non-constant coefficients. We complement their results by computing the Hodge type of Eisenstein cohomology. The last section combines all results and contains the proofs of the above theorems.

Notations: X is a fixed Hilbert modular variety of dimension $n \geq 2$. X^* is the Baily-Borel compactification of X , \bar{X} is a smooth toroidal compactification of X with the boundary divisor $S = \bar{X} - X$. Let h be the number of cusps, which is the cardinality of the set $X^* - X$. Let $h^{n,0}(\bar{X})$ be the geometric genus of \bar{X} . For any n -tuple $m = (m_1, \dots, m_n) \in \mathbb{N}_0^n$, put $|m| = \sum_{i=1}^n m_i$, and generally for any subset $I \subset \{1, \dots, n\}$, $|m_I| = \sum_{i \in I} m_i$. The cohomology group $H^k(X, \mathbb{V}_m)$ has a natural real MHS with weights $\geq |m| + k$. Define

$$h_k^{p,q} := \dim Gr_F^p Gr_{\bar{F}}^q Gr_{p+q}^W H^k(X, \mathbb{V}_m),$$

as Hodge numbers of the mixed Hodge structure. For $0 \leq k \leq 2n$, $IH^k(X^*, \mathbb{V})$ is the middle perversity intersection cohomology of \mathbb{V} over X (see §4), and $H_{\text{Eis}}^k(X, \mathbb{V})$ is the Eisenstein cohomology of \mathbb{V} over X (see §5). Over \bar{X} , there are a set of basic line bundles $\{\mathcal{L}_i\}_{1 \leq i \leq n}$ (see §3), which are good extensions to \bar{X} of locally homogenous line bundles over X in the sense of Mumford (see [20]).

2. PRELIMINARIES AND DELIGNE-SAITO'S MIXED HODGE STRUCTURE ON THE COHOMOLOGY GROUP

Let $F \subset \mathbb{R}$ be a totally real number field of degree $n \geq 2$ over \mathbb{Q} , with the set of real embeddings $\text{Hom}_{\mathbb{Q}}(F, \mathbb{R}) = \{\sigma_1 = id, \dots, \sigma_n\}$. Let \mathcal{O}_F be the integer ring of F and \mathcal{O}_F^* be the unit group of \mathcal{O}_F . For an element $a \in F$ one puts $a^{(i)} = \sigma_i(a)$, the i -th Galois conjugate of a . Let $G = R_{F|\mathbb{Q}}SL_2$ be the \mathbb{Q} -algebraic group obtained by Weil restriction. The set of real points $G(\mathbb{R})$ of G is identified with $G_1 \times \dots \times G_n$, where each G_i is a copy of $SL(2, \mathbb{R})$. The subset $G(\mathbb{Q}) \subset G(\mathbb{R})$ is then given by

$$\left\{ \left(\begin{pmatrix} a^{(1)} & b^{(1)} \\ c^{(1)} & d^{(1)} \end{pmatrix}, \dots, \begin{pmatrix} a^{(n)} & b^{(n)} \\ c^{(n)} & d^{(n)} \end{pmatrix} \right) \in G_1 \times \dots \times G_n \mid a, b, c, d \in F \right\}.$$

Now let $\mathbb{H}^n = \mathbb{H}_1 \times \dots \times \mathbb{H}_n$ be the product of n copies of the upper half plane with coordinates

$$z = (z_1 = x_1 + iy_1, \dots, z_l = x_l + iy_l, \dots, z_n = x_n + iy_n).$$

The group $G(\mathbb{R})$ acts on \mathbb{H}^n by a product of linear fractional transformation. Namely, for $g = \left(g_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \dots, g_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \right) \in G(\mathbb{R})$ and $z \in \mathbb{H}^n$, the action is given by $g \cdot z = (g_1 \cdot z_1, \dots, g_n \cdot z_n)$, where $g_i \cdot z_i = (a_i z_i + b_i)(c_i z_i + d_i)^{-1}$. The action is transitive and the isotropy subgroup of $G(\mathbb{R})$ at the base point $z_0 = (i, \dots, i)$ is $K = SO(2)^n$, a maximally compact subgroup. K acts on G by right multiplication and one identifies the set G/K of left cosets naturally with \mathbb{H}^n . Let $\Gamma \subset G(\mathbb{Q})$ be a torsion-free subgroup which is commensurable with $G(\mathbb{Z}) \subset G(\mathbb{Q})$. It is called a *Hilbert modular group* in this paper and will be fixed throughout. By the theorem of Baily-Borel, the quotient space $X_\Gamma := \Gamma \backslash G(\mathbb{R})/K$ is naturally a smooth quasi-projective variety, which is called the *Hilbert modular variety* for Γ . As Γ is fixed, we shall denote X_Γ simply by X . The topological space X is non-compact and admits several natural compactifications. The Baily-Borel compactification X^* of X is obtained by adding a number of *cusps* as a set (see [8]) and has the structure of a projective variety by Baily-Borel. It is however singular, and admits a natural family of resolutions of singularities, the so-called smooth toroidal compactifications (see [1] for general locally symmetric varieties and [6] more details for Hilbert modular varieties). Let \bar{X} be such a smooth compactification, and let $S = \bar{X} - X$ be the divisor at infinity, which has simple normal crossings. In addition, we also use the Borel-Serre compactification X^\sharp in §5. It is a smooth compact manifold with boundary, which contains X as the interior open subset. The boundary $\partial X^\sharp = X^\sharp - X$ has h components in total, with each component an $(S^1)^n$ -bundle over $(S^1)^{n-1}$.

For $A \subset \mathbb{C}$ a \mathbb{Q} -subalgebra, we define an A -local system over X as a locally constant sheaf of free A -modules of finite rank with respect to the analytic topology on X . Let 0 be the point of X given by the Γ -equivalence class of $z_0 \in \mathbb{H}^n$. Then it is well-known that an A -local system over X corresponds to a representation $\pi_1(X, 0) \rightarrow GL(A)$. In this paper A is either \mathbb{Q} , \mathbb{R} or \mathbb{C} . Since $\pi_1(X, 0)$ is naturally identified with Γ , by the super-rigidity theorem of Margulis, equivalence classes of complex local systems over X are in one-to-one correspondence with equivalence classes of finite dimensional complex representations of $G(\mathbb{R})$. Let \mathbb{V} be the complex local system corresponding to the irreducible representation $\rho : G(\mathbb{R}) \rightarrow GL(V)$. By Schur's lemma, there exists an n -tuple $m = (m_1, \dots, m_n)$ of non-negative integers and n copies V_i of \mathbb{C}^2 for $i = 1, \dots, n$ such that $\rho = \rho_{m_1} \otimes \dots \otimes \rho_{m_n}$, where for each i , $\rho_{m_i} : G_i \rightarrow GL(S^{m_i} V_i)$ is isomorphic to the m_i -th symmetric power of the standard complex representation of $SL(2, \mathbb{R})$. A local system \mathbb{V} is called *regular* if ρ is a regular representation, i.e., its highest weight is contained in the interior of the Weyl chamber. It is clear that $\mathbb{V} = \mathbb{V}_m$ is regular if and only if each m_i in the above is positive. To summarize, for each $m \in \mathbb{N}_0^n$, there is a unique complex

local system \mathbb{V}_m over X up to isomorphism, and any complex local system over X is a finite direct sum of such. For $(0, \dots, m_i, \dots, 0)$ we denote the corresponding local system by \mathbb{V}_{i, m_i} . So $\mathbb{V}_m = \mathbb{V}_{1, m_1} \otimes \dots \otimes \mathbb{V}_{n, m_n}$. The complex local system $\mathbb{V} = \mathbb{V}_m$ is the complexification of a natural real local system $\mathbb{V}_{\mathbb{R}}$. Moreover, $\mathbb{V}_{\mathbb{R}}$ is naturally an \mathbb{R} -PVHS, which is a special case considered by Zucker (see [32]). This can be seen as follows. Let

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

be the standard basis of \mathbb{C}^2 on which $SL(2, \mathbb{R})$ acts by matrix multiplication. Then $\mathbb{R}^2 = \mathbb{R}e_1 + \mathbb{R}e_2 \subset \mathbb{C}^2$ is an invariant \mathbb{R} -structure. Define a symplectic form ω on \mathbb{R}^2 such that $\{e_1, e_2\}$ is the symplectic basis for ω . Let $\mathbb{C}^{1,0} = \mathbb{C}\{e_1 + ie_2\}$ and $\mathbb{C}^{0,1} = \mathbb{C}\{e_1 - ie_2\}$. Then the decomposition $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}^{1,0} \oplus \mathbb{C}^{0,1}$ defines a polarized weight one Hodge structure on \mathbb{R}^2 , and this decomposition has the special property that it is also the eigen-decomposition of the induced action of $SO(2)$ by restriction. For each $1 \leq i \leq n$, the i -th factor of $G(\mathbb{R})$ acts on V_i via the standard representation and trivially on any other factor. Applying the foregoing construction on \mathbb{C}^2 to V_i , one obtains a polarized weight one Hodge structure on the fiber of the constant bundle $\mathbb{H}^n \times V_i$ at $z_0 \in \mathbb{H}^n$. By using the homogeneity property of \mathbb{H}^n , one defines a \mathbb{R} -PVHS on the constant bundle $\mathbb{H}^n \times V_i$, and it descends to a \mathbb{R} -PVHS on $\mathbb{V}_{i,1}$ over X (see §4 in [32]). Taking the m_i -th symmetric power, one obtains a \mathbb{R} -PVHS of weight m_i on \mathbb{V}_{i, m_i} , and further by taking tensor products one obtains a \mathbb{R} -PVHS of weight $|m| = \sum_{i=1}^n m_i$ on \mathbb{V}_m as claimed. It is clear that $\mathbb{V}_{\mathbb{R}}$ is in fact defined over $F \subset \mathbb{R}$, and is even defined over \mathbb{Q} if (and only if) $m_1 = \dots = m_n$ holds.

Now we consider an even more general setting. Let M be a quasi-projective manifold of dimension d and $(\mathbb{W}_{\mathbb{R}}, \nabla, F^\cdot)$ a \mathbb{R} -PVHS over M of weight n . Let \bar{M} be a smooth, projective compactification of M such that $S = \bar{M} - M$ is a simple normal crossing divisor. For simplicity of exposition, we assume that the local monodromy around each irreducible component of S is unipotent (it is quasi-unipotent in general). Put $\mathbb{W}_{an} = \mathbb{W}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathcal{O}_{M_{an}}$, where $\mathcal{O}_{M_{an}}$ is the sheaf of germs of holomorphic functions on M . Deligne's canonical extension (see Ch II, §4 in [4]) gives a unique extended vector bundle $\bar{\mathbb{W}}_{an}$ of \mathbb{W}_{an} over \bar{M} , together with a flat logarithmic connection $\bar{\nabla} : \bar{\mathbb{W}}_{an} \rightarrow \bar{\mathbb{W}}_{an} \otimes \Omega_{\bar{M}_{an}}^1(\log S)$. Using this we obtain the logarithmic de-Rham complex $\Omega_{\log}^*(\bar{\mathbb{W}}_{an}, \bar{\nabla})$. Schmid's Nilpotent orbit theorem implies that the Hodge filtration F^\cdot extends to a filtration \bar{F}^\cdot of holomorphic subbundles of $\bar{\mathbb{W}}_{an}$ as well (see §4 in [27]). By GAGA, the extended holomorphic objects over \bar{M} are in fact algebraic. One defines a Hodge filtration

on the logarithmic de-Rham complex by $F^r \Omega_{\log}^*(\bar{\mathbb{W}}_{an}, \bar{\nabla}) = \Omega_M^*(\log S) \otimes \bar{F}^{r-*}$, which is a subcomplex by Griffiths transversality. After Saito (see [21]-[24]), there is a naturally defined weight filtration W_\cdot on the logarithmic de-Rham complex such that the triple $(\Omega_{\log}^*(\bar{\mathbb{W}}_{an}, \bar{\nabla}), W_\cdot, F^\cdot)$ is a mixed Hodge complex (see Appendix A in [7]). By Scholie 8.1.9 (ii) in [5], this gives rise to a real MHS with weights $\geq k+n$ on $H^k(M, \mathbb{W}_{\mathbb{R}})$. When $\mathbb{W}_{\mathbb{R}}$ is constant, this MHS coincides with the one defined in §3.2, [5] by Deligne. It is this MHS that we intend to understand properly in the case of Hilbert modular varieties.

After taking the graded object associated to the pair $(\bar{\mathbb{W}}_{an}, \bar{\nabla})$ with respect to the filtration \bar{F}^\cdot , one obtains the logarithmic Higgs bundle $(E = \bigoplus_{p+q=n} E^{p,q}, \theta = \bigoplus_{p+q=n} \theta^{p,q})$. Again, by Griffiths transversality it satisfies $\theta^{p,q} : E^{p,q} \rightarrow E^{p-1,q+1} \otimes \Omega_M^1(\log S)$. The Higgs field is integrable, i.e., $\theta \wedge \theta = 0$, and so one can form the logarithmic Higgs complex $\Omega_{\log}^*(E, \theta)$ (see [29] for the compact case, and the definition for the current case is similar). Note that the sheaves and the differentials in this complex are algebraic. The hypercohomology of this complex is called *logarithmic Higgs cohomology*. One observes that the logarithmic Higgs complex is a direct sum $\bigoplus_{P=0}^{d+n} \Omega_P^*(E, \theta)$ of subcomplexes, where $\Omega_P^l(E, \theta) = E^{P-l, m-P+l} \otimes \Omega_M^l(\log S)$. We define $\mathcal{C}^{P,l}(E, \theta)$ to be the l -th cohomology sheaf of the subcomplex $\Omega_P^*(E, \theta)$. The relation of logarithmic Higgs cohomology with Saito's MHS is given by the following

Proposition 2.1. *Let $\mathbb{W} = \mathbb{W}_{\mathbb{R}} \otimes \mathbb{C}$ and F^\cdot be the Hodge filtration on $H^k(M, \mathbb{W})$ as above. For $0 \leq k \leq d$, one has a natural isomorphism*

$$Gr_F^P H^k(M, \mathbb{W}) \simeq \mathbb{H}^k(\bar{M}, \Omega_P^*(E, \theta)).$$

Proof. This is in fact a direct consequence of the E_1 -degeneration of the Hodge filtration on $\Omega_{\log}^*(\bar{\mathbb{W}}_{an}, \bar{\nabla})$ by Scholie 8.1.9 (v) in [5]. \square

When \mathbb{W} is constant, the Higgs field is trivial. Therefore the logarithmic Higgs cohomology is just the usual sheaf cohomology. In another important case the above result can also be improved:

Proposition 2.2. *Assume that M is a smooth arithmetic quotient of a hermitian symmetric domain and \bar{M} a smooth toroidal compactification. Moreover assume that $\mathbb{W}_{\mathbb{R}}$ is locally homogenous (see [32]). Then one has a natural isomorphism*

$$Gr_F^P H^k(M, \mathbb{W}) \simeq \bigoplus_{l=0}^d H^{k-l}(\bar{M}, \mathcal{C}^{P,l}(E, \theta)).$$

Proof. The proof is based on the one of Proposition 5.19 in [32] for compact M . Let $\Omega^*(E^0, \theta^0)$ be the Higgs complex of \mathbb{W} over X , which is equal to the restriction of the logarithmic Higgs

complex $\Omega^*(E, \theta)$ over \bar{X} to X . It suffices to show that the logarithmic Higgs complex is the good extension of the Higgs complex in the sense of Mumford. First we consider the terms in the complex. Each term is a tensor product of a wedge power of $\Omega_{\bar{M}}^1(\log S)$ and the Deligne-Schmid extension of $E^0 = E|_M$. Proposition 3.4 in [20] shows that $\Omega_{\bar{M}}^1(\log S)$ is the good extension of Ω_M^1 . The estimates of the Hodge metric, which is the invariant metric on E^0 up to scalar, provided by Theorem 5.21 in [3], shows that the sections of E are at most of logarithmic growth around S with respect to the Hodge metric. By the characterization of the good extension in Proposition 1.3, [20], one concludes that E is the good extension of E^0 . We also have to consider the bundle homomorphism: we may choose a local basis of group invariant sections for E^0 and conclude that θ^0 is a constant morphism in this basis, since, by homogeneity, θ^0 is determined by its value at one point. In the extended group invariant basis also θ is constant on \bar{M} . This argument implies that the l -th cohomology sheaf $\mathcal{C}^{P,l}(E, \theta)$ of $\Omega_P^*(E, \theta)$ is the good extension of l -th cohomology of $\Omega_P^*(E^0, \theta^0)$ to \bar{X} , which is a direct summand of $\Omega_P^l(E, \theta)$. Moreover, the inclusion $\bigoplus_l \mathcal{C}^{P,l}(E, \theta)[-l] \hookrightarrow \Omega_P^*(E, \theta)$ is a quasi-isomorphism. This completes the proof. \square

3. LOGARITHMIC HIGGS COHOMOLOGY OVER HILBERT MODULAR VARIETY

Let $\mathbb{V} = \mathbb{V}_m$ (resp. \mathbb{V}_{i,m_i}) be an irreducible complex local system over the Hilbert modular variety X . After a possible finite étale base change of X , the local monodromies of \mathbb{V} at infinity can be made unipotent, so we assume this from now on. This assumption can be removed once a result is unaffected by finite étale base change. Let (E_m, θ_m) (resp. $(E_{i,m_i}, \theta_{i,m_i})$) be the resulting logarithmic Higgs bundle over \bar{X} by the construction in §2. It is clear that the construction is compatible with direct sums or tensor products, and hence

$$(E_m, \theta_m) = (E_{1,m_1}, \theta_{1,m_1}) \otimes \cdots \otimes (E_{n,m_n}, \theta_{n,m_n}).$$

In this section the logarithmic Higgs cohomology of (E_m, θ_m) will be determined.

We start with a basic property of the set $\{(E_{i,1}, \theta_{i,1}), 1 \leq i \leq n\}$ of logarithmic Higgs bundles.

Proposition 3.1. *For each i , $E_{1,i} = \mathcal{L}_i \oplus \mathcal{L}_i^{-1}$. There exists a natural isomorphism $\Omega_{\bar{X}}^1(\log S) \simeq \bigoplus_{i=1}^n \mathcal{L}_i^2$ such that $\theta_{1,i}^{1,0} : \mathcal{L}_i \rightarrow \mathcal{L}_i^{-1} \otimes \Omega_{\bar{X}}^1(\log S)$ is the composition of the tautological maps:*

$$\mathcal{L}_i \xrightarrow{\simeq} \mathcal{L}_i^{-1} \otimes \mathcal{L}_i^2 \hookrightarrow \mathcal{L}_i^{-1} \otimes \Omega_{\bar{X}}^1(\log S).$$

Proof. This follows from the fact that the period map for Hilbert modular varieties is an embedding together with uniqueness of the Mumford extension. \square

Consider the following situation: Let D_i , $i = 1, 2$ be two hermitian symmetric domains, and $(\mathbb{V}_i, \nabla_i, F_i)$ a homogenous VHS over D_i (see [32]). Let (E_i, θ_i) be the (analytic) Higgs bundle, which is obtained by taking the grading of the pair $(\mathbb{V}_{i,an}, \nabla_i)$ with respect to F_i . Similarly as before, one forms the Higgs complex $\Omega^*(E_i, \theta_i)$ over D_i . Now put $D = D_1 \times D_2$ with two projections p_i , $i = 1, 2$. By abuse of notation we denote the pull-back of (E_i, θ_i) via p_i to D by the same symbol. Put $(E, \theta) = (E_1, \theta_1) \otimes (E_2, \theta_2)$. It is straightforward to verify the Künneth decomposition of the Higgs complex

$$\Omega_P^*(E, \theta) = \bigoplus_{P_1+P_2=P} \Omega_{P_1}^*(E_1, \theta_1) \otimes \Omega_{P_2}^*(E_2, \theta_2),$$

and then the Künneth decomposition of the cohomology sheaves

$$\mathcal{C}^{P,l}(E, \theta) = \bigoplus_{\substack{P_1+P_2=P, \\ l_1+l_2=l}} \mathcal{C}^{P_1,l_1}(E_1, \theta_1) \otimes \mathcal{C}^{P_2,l_2}(E_2, \theta_2).$$

Summarizing the above discussion we obtain the following

Proposition 3.2. *Let D_i for $1 \leq i \leq n$ be hermitian symmetric domains, and \mathbb{V}_i a homogenous VHS over D_i with the corresponding Higgs bundle (E_i, θ_i) . Put $D = D_1 \times \cdots \times D_n$, and $(E, \theta) = (E_1, \theta_1) \otimes \cdots \otimes (E_n, \theta_n)$, which corresponds to $\mathbb{V} = \mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_n$ over D . Then one has the following Künneth formula for cohomology sheaves of $\Omega^*(E, \theta)$:*

$$\mathcal{C}^{P,l}(E, \theta) = \bigoplus_{\substack{P_1+\cdots+P_n=P, \\ l_1+\cdots+l_n=l}} \mathcal{C}^{P_1,l_1}(E_1, \theta_1) \otimes \cdots \otimes \mathcal{C}^{P_n,l_n}(E_n, \theta_n).$$

Let us go back to the discussion about the cohomology sheaves of $\Omega_{\log}^*(E_m, \theta_m)$ over \tilde{X} .

Proposition 3.3. *For any subset $I \subset \{1, \dots, n\}$ define I^c to be the complement of I , $|m_I| = \sum_{i \in I} m_i$ and $\mathcal{C}_I = \bigotimes_{i \in I} \mathcal{L}_i^{m_i+2} \otimes \bigotimes_{i \in I^c} \mathcal{L}_i^{-m_i}$. Then one has the formula*

$$\mathcal{C}^{P,l}(E_m, \theta_m) = \bigoplus_{\substack{I \subset \{1, \dots, n\}, \\ |m_I|+|I|=P, |I|=l}} \mathcal{C}_I.$$

Proof. By the proof of Proposition 2.2, each $\mathcal{C}^{P,l}(E_m, \theta_m)$ is the good extension of the corresponding cohomology sheaf for the Higgs complex, the restriction of $\Omega_P^*(E_m, \theta_m)$ to X . By the homogeneity of the Higgs complex, it suffices to carry out the computation at 0, or equivalently the computation of the pull-back Higgs complex over \mathbb{H}^n at z_0 . For that, one applies Proposition 3.2 for $D_i = \mathbb{H}$, $1 \leq i \leq n$. For each i , the computation over D_i is easy and the details are therefore omitted. \square

The above proposition and Proposition 2.2 imply the following

Corollary 3.4. *Let $0 \leq k \leq 2n$ and F^\cdot be the Hodge filtration on $H^k(X, \mathbb{V}_m)$. For each $0 \leq P \leq |m| + k$, one has a natural isomorphism*

$$Gr_F^P H^k(X, \mathbb{V}_m) \simeq \bigoplus_{\substack{I \subset \{1, \dots, n\}, \\ |m_I| + |I| = P}} H^{k-|I|}(\bar{X}, \mathcal{C}_I).$$

We conclude this section with two vanishing results of global sections of \mathcal{C}_I , which contribute to a priori information about the sheaf cohomologies in the above corollary. Note that the line bundle \mathcal{C}_I is of the form $\bigotimes_{i=1}^n \mathcal{L}_i^{s_i}$ for an n -tuple of integers (s_1, \dots, s_n) . The first vanishing result is easy to prove and valid for a product of compactifications of modular curves, which can be viewed as the degenerate situation of Hilbert modular varieties. As this result will not be applied in the following sections, its proof is omitted.

Proposition 3.5. *For an n -tuple (s_1, \dots, s_n) of non-zero integers one has the vanishing*

$$H^0(\bar{X}, \bigotimes_{i=1}^n \mathcal{L}_i^{s_i}) = 0$$

if one of the s_i is negative.

The second vanishing result relies essentially on the irreducibility of the arithmetic lattice Γ , and is of differential geometric nature:

Proposition 3.6. *For an n -tuple $(s_1, \dots, s_n) \in \mathbb{N}_0^n \setminus \{0, \dots, 0\}$ one has the vanishing*

$$H^0(X, \bigotimes_{i=1}^n \mathcal{L}_i^{s_i}|_X) = 0$$

if one of the entries s_i is zero.

We first recall the following

Definition 3.7. Let M be a complex manifold and (F, h) a hermitian holomorphic vector bundle over M . Let Θ be the curvature tensor of (F, h) . (F, h) is said to be semi-positive (in the sense of Griffiths) at the point $x \in M$, if for any non-zero tangent vector $v \in T_{X,x}$ and any non-zero vector $e \in F_x$, $\Theta(e, \bar{e}, v, \bar{v}) \geq 0$. It is said to be properly semi-positive if furthermore for certain non-zero vectors e_0 and v_0 and has $\Theta(e_0, \bar{e}_0, v_0, \bar{v}_0) = 0$.

The significance of the notion *properly semi-positive* in the case of locally hermitian symmetric domains lies in the following theorem due to N. Mok, as a direct consequence of his metric rigidity theorem on irreducible locally homogenous bundles:

Theorem 3.8 (Mok). *Let D be a hermitian symmetric domain and Γ a torsion-free discrete subgroup of $G = \text{Aut}^0(D)$ such that the quotient $M = \Gamma \backslash D$ is irreducible (i.e. not a product*

of two complex manifolds of positive dimensions) and has finite volume with respect to the canonical metric. Let (F, h) be a non-trivial irreducible locally homogenous vector bundle over M with an invariant hermitian metric h . If (F, h) is properly semi-positive at one point (and hence for all points), then $H^0(X, F) = 0$.

Proof. See §2.2 Theorem 1 and §2.3 in Chapter 10, [19]. \square

The following proposition gives a sufficient condition for a semi-positive locally homogenous bundle being properly semi-positive.

Proposition 3.9. *Let M be as above, and (F, h) a semi-positive locally homogenous bundle over M . If there is a Higgs bundle (E, θ) corresponding to a locally homogenous VHS \mathbb{W} over M such that $F \subset E^{p,q}$ and there exist non-zero vectors $e \in F_0$ and $v \in T_{X,0}$ such that $\theta_v(e) = 0$, then (F, h) is properly semi-positive.*

Proof. Because F is an irreducible component of $E^{p,q}$, the second fundamental form vanishes (see Chapter I in [15]). So we can use Griffiths' curvature formula (see Lemma 7.18 in [27] for example) for $E^{p,q}$ to calculate the curvature of F :

$$\Theta(e, e') = (\theta(e), \theta(e')) - (\theta^\dagger(e), \theta^\dagger(e')) \quad \forall e, e' \in E^{p,q}.$$

By assumption, $\theta_v(e) = 0$. So

$$\Theta(e, \bar{e}, v, \bar{v}) = (\theta_v(e), \theta_v(e)) - (\theta_v^\dagger(e), \theta_v^\dagger(e)) = -(\theta_v^\dagger(\bar{e}), \theta_v^\dagger(\bar{e})) \leq 0.$$

Since F is semi-positive, $\Theta(e, \bar{e}, v, \bar{v}) \geq 0$. Therefore $\Theta(e, \bar{e}, v, \bar{v}) = 0$. \square

Back to the proof of Proposition 3.6:

Proof. Take $m = (s_1, \dots, s_n)$. By the discussion at the beginning of this section, it is clear that $\bigotimes_{i=1}^n \mathcal{L}_i^{s_i}|_X$ is the first Hodge bundle, namely the $E^{|m|,0}$ part of the corresponding Higgs bundle to \mathbb{W}_m . So it is semi-positive. Assume $s_1 = 0$ without loss of generality. Let e_i be a local section of \mathcal{L}_i at 0. Via the natural projection map, X and \mathbb{H}^n are analytically local isomorphic to each other. A generator of the tangent subspace $T_{\mathbb{H}^n, i} \subset T_{\mathbb{H}^n, z_0}$ gives rise to a tangent vector v of $T_{X,0}$. By Proposition 3.1, the Higgs field $\theta_{1,i}$ along the direction v acts trivially on e_i for $2 \leq i \leq n$. For $\theta_m = \bigotimes_{i=2}^n \theta_{1,i}^{s_i}$, it follows that for the local section $e = \bigotimes_{i=2}^n e_i^{s_i}$ of $\bigotimes_{i=1}^n \mathcal{L}_i^{s_i}$, $\theta_{m,v}(e) = 0$. By Proposition 3.9 and Theorem 3.8, the proposition follows. \square

4. INTERSECTION COHOMOLOGY OF LOCAL SYSTEMS OVER HILBERT MODULAR VARIETY AND THE PURE HODGE STRUCTURE

We may resume the general setting discussed in the last part of §2 and quote the following fundamental result, which was conjectured by Zucker [33]:

Theorem 4.1 (Looijenga [17], Saper-Stern [30]). *Let M be a smooth arithmetic quotient of hermitian symmetric domain with M^* the Baily-Borel compactification. Let \mathbb{W} be a locally homogenous VHS over M . Let g (resp. h) be the group invariant metric on M (resp. \mathbb{W}) (unique up to constant), and let $H_{(2)}^k(M, \mathbb{W})$ be the L^2 -cohomology group of degree k with coefficients in \mathbb{W} with respect to the above metrics. Let $IH^k(M^*, \mathbb{W})$ be the k -th (middle perversity) intersection cohomology. Then one has natural isomorphism*

$$IH^k(M^*, \mathbb{W}) \simeq H_{(2)}^k(M, \mathbb{W}).$$

The case of Hilbert modular varieties has already been verified in [33].

Theorem 4.2 (Zucker [32]). *Let M, M^* be as above. If \mathbb{W} underlies a locally homogenous \mathbb{R} -PVHS $\mathbb{W}_{\mathbb{R}}$ of weight m , $IH^k(M^*, \mathbb{W}_{\mathbb{R}})$ carries a natural real pure Hodge structure of weight $k + m$.*

In this section we shall describe the Hodge decomposition of $IH^k(X^*, \mathbb{V}_{\mathbb{R}})$ and give a formula for the Hodge numbers. For a subset $I \subset \{0, \dots, n\}$, we denote by \mathcal{C}_I^0 the restriction of \mathcal{C}_I to X , which is a locally homogenous line bundle over X , and by $H_{(2)}^*(X, \mathcal{C}_I^0)$ the L^2 -Dolbeault cohomology of \mathcal{C}_I^0 with respect to the *invariant* metric g of X and the group invariant metric h on \mathcal{C}_I^0 . That is, it is the cohomology of the complex $(A_{(2)}^{0,*}(X, \mathcal{C}_I^0), \bar{\partial})$, where $A_{(2)}^{0,i}(X, \mathcal{C}_I^0)$ is the space of L^2 -integrable smooth $(0, i)$ -forms with coefficients in \mathcal{C}_I^0 over X . Let $\mathfrak{h}_{\bar{\partial}, (2)}^i(X, \mathcal{C}_I^0) \subset A_{(2)}^{0,i}(X, \mathcal{C}_I^0)$ be the subspace of L^2 - $\bar{\partial}$ -harmonic forms. By the L^2 -harmonic theory, one has a natural isomorphism $H_{(2)}^i(X, \mathcal{C}_I^0) \simeq \mathfrak{h}_{\bar{\partial}, (2)}^i(X, \mathcal{C}_I^0)$.

Theorem 4.3. *Let $I \subset \{1, \dots, n\}$ be a subset. Then the following statement holds:*

$$\dim H_{(2)}^i(X, \mathcal{C}_I^0) = \begin{cases} 0, & i \neq n - |I|; \\ \dim H_{(2)}^0(X, \bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2}|_X), & i = n - |I|. \end{cases}$$

In order to prove it, we need to employ the full machinery of L^2 -harmonic theory. Let $A_{(2)}^n(X, \mathbb{V}_m)$ (resp. $A_{(2)}^{p,q}(X, \mathbb{V}_m)$) be the space of smooth L^2 - n -forms (resp. type (p, q) -forms) on X with coefficients in \mathbb{V}_m . Denote the subspace of L^2 -harmonic forms by

$$\mathfrak{h}_{(2)}^n(X, \mathbb{V}_m) = \{\alpha \in A_{(2)}^n(X, \mathbb{V}_m) \mid \square_D \alpha = 0\},$$

similarly the subspace of L^2 -harmonic (p, q) -forms by $\mathfrak{h}_{(2)}^{p,q}(X, \mathbb{V}_m)$. By Zucker [32], one has natural isomorphisms

$$IH^n(X^*, \mathbb{V}_m) \simeq \mathfrak{h}_{(2)}^n(X, \mathbb{V}_m) = \bigoplus_{p+q=n} \mathfrak{h}_{(2)}^{p,q}(X, \mathbb{V}_m).$$

By abuse of notation, in the following we denote again by (E_m, θ_m) the restriction of (E_m, θ_m) to X , which is a Higgs bundle over X . We consider the cohomology sheaf of the Higgs complex of (E_m, θ_m) over X at the i -th place:

$$E_m \otimes \Omega_X^{i-1} \xrightarrow{\theta_m^{i-1}} E_m \otimes \Omega_X^i \xrightarrow{\theta_m^i} E_m \otimes \Omega_X^{i+1},$$

and denote by $\theta_m^{*,i}$ the adjoint of θ_m^i with respect to the Hodge metric on E_m and the invariant metric on X . We have the following

Lemma 4.4. *The i -th cohomology sheaf of the Higgs complex can be characterized by those sections of $E_m \otimes \Omega_X^i$ satisfying the equations $\theta_m^i = \theta_m^{*,i-1} = 0$.*

Proof. Let $\mathcal{C} = \frac{\text{Ker } \theta_m^i}{\text{Im } \theta_m^{i-1}}$ be the cohomology sheaf. By the homogeneity, one has the holomorphic and metric decomposition of Hermitian vector bundles

$$\text{Ker } \theta_m^i = \mathcal{C} \oplus \text{Im } \theta_m^{i-1}.$$

So $\mathcal{C} = \text{Ker } \theta_m^i \cap (\text{Im } \theta_m^{i-1})^\perp$. On the other hand, one has clearly that $\text{Ker } \theta_m^{*,i-1} = (\text{Im } \theta_m^{i-1})^\perp$. The lemma follows. \square

Lemma 4.5. *For each I , one has the inclusion $\mathfrak{h}_{\partial,(2)}^i(X, \mathcal{C}_I^0) \subset \mathfrak{h}_{(2)}^{|I|,i}(X, \mathbb{V}_m)$.*

Proof. This follows from (5.22) and (5.14) in [32]. We need to explain the notation. By Proposition 3.3, \mathcal{C}_I^0 is a direct summand of the $|I|$ -th cohomology sheaf of the Higgs subcomplex $\Omega_{|m_I|+|I|}^*(E_m, \theta_m)$. By (5.22) and (5.14) in [32], it follows that

$$\mathfrak{h}_{\partial,(2)}^i(X, \mathcal{C}_I^0) \subset \mathfrak{h}_{(2)}^{|I|,i}(X, \mathbb{V}_m).$$

\square

Let $I = (i_1, \dots, i_p), J = (j_1, \dots, j_q)$ be two multi-indices with $1 \leq i_1 < \dots < i_p \leq n$ and $1 \leq j_1 < \dots < j_q \leq n$. Put $dz_I = dz_{i_1} \wedge \dots \wedge dz_{i_p}$ and $\overline{dz_J} = d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$. Let $\mathfrak{h}_{(2)}(I, J; \mathbb{V}_m)$ be the subspace of $\mathfrak{h}_{(2)}^{p,q}(X, \mathbb{V}_m)$ consisting of those elements whose pull-back to \mathbb{H}^n are of form $f_{I,J} dz_I \wedge \overline{dz_J}$. The following lemma is the L^2 -analogue of Proposition 1.2 in [18] and its proof holds verbatim for L^2 -harmonic forms.

Lemma 4.6. *One has an orthogonal decomposition into a direct sum of subspaces of L^2 -harmonic forms for each (p, q) :*

$$\mathfrak{h}_{(2)}^{p,q}(X, \mathbb{V}_m) = \bigoplus_{\substack{I, J \subset \{1, \dots, n\}, \\ |I|=p, |J|=q}} \mathfrak{h}_{(2)}(I, J; \mathbb{V}_m)$$

The next proposition gives the L^2 -analogue of Proposition 4.1-4.3 in [18] in the case of non-trivial local systems.

Proposition 4.7. *Let I, J be as above. If $I \cup J$ is not a permutation of $\{1, \dots, n\}$, then $\mathfrak{h}_{(2)}(I, J; \mathbb{V}_m) = 0$.*

Proof. Let $\alpha = \alpha_{I, \bar{J}} \in \mathfrak{h}_{(2)}(I, J; \mathbb{V}_m)$. We prove the statement by induction on $q = |J|$. When $q = 0$, $\alpha \in A^{p,0}(X, \mathbb{V}_m) = A^{0,0}(X, E_m \otimes \Omega_X^p)$. Because α is D -harmonic and equivalently D'' -harmonic, one has $\bar{\partial}(\alpha) = 0$, $\theta(\alpha) = 0$ and $\theta^*(\alpha) = 0$ (see Corollary 3.20 in [32]). This implies that α is a global holomorphic section of $E_m \otimes \Omega_X^p$ and by Lemma 4.4 it is even a global section of the cohomology sheaf of the Higgs complex at the $p = |I|$ -th place. Thus it must be a global section of $\mathcal{C}_I^0 = \bigotimes_{i \in I} \mathcal{L}_i^{m_i+2} \otimes \bigotimes_{i \in I^c} \mathcal{L}_i^{-m_i}|_X$ by consideration of $\{I, J; \mathbb{V}_m\}$ -type.

Case 1: $I \neq \{1, \dots, n\}$ and $m_i = 0$ for all $i \in I^c$. In this case, \mathcal{C}_I^0 is of the type in Proposition 3.6. Since $\alpha \in \mathfrak{h}_{\bar{\partial}, (2)}^0(X, \mathcal{C}_I^0) \subset H^0(X, \mathcal{C}_I^0)$, $\alpha = 0$ by Proposition 3.6.

Case 2: $m_i \neq 0$ for certain $i \in I^c$. In this case, we consider the decomposition of the differential operator D over the space $A_{(2)}^p(X, \mathbb{V}_m)$ (see §1 and §4 in [29]):

$$D = D' + D'' = d' + d''; \quad D' = \partial + \bar{\theta}, \quad D'' = \bar{\partial} + \theta, \quad d' = \partial + \theta, \quad d'' = \bar{\partial} + \bar{\theta}.$$

$\square_D(\alpha) = 0$ implies that $d''(\alpha) = 0$. Since $\bar{\partial}\alpha = 0$, it follows that $\bar{\theta}(\alpha) = 0$. Now we take a smooth open neighborhood U of $0 \in X$ with the local coordinates $\{z_1, \dots, z_n\}$ and e be a holomorphic basis of the line bundle \mathcal{C}_I^0 on U . Write $\alpha = f(z_1, \dots, z_n)e$ over U . Assuming the following claim, one has

$$\bar{\theta}(\alpha) = \bar{\theta}(f(z)e) = \bar{f}(z)\bar{\theta}(e) = 0,$$

and then $f = 0$. So $\alpha = 0$. The following claim is a direct consequence of Proposition 3.1.

Claim 4.8. One has $\bar{\theta}(e) \neq 0$, where e is as above and

$$\bar{\theta} : A_{(2)}^{(p,0)}(E^{|m_I|, |m| - |m_I|}) \rightarrow A_{(2)}^{(p,1)}(E^{|m_I|+1, |m| - |m_I| - 1}).$$

Proof. Let $\theta_i = \theta_{\partial_{z_i}}$ be the Higgs field along the tangent direction ∂_{z_i} . Let e_i (resp. e_i^*) be a local basis of \mathcal{L}_i (resp. \mathcal{L}_i^{-1}) over U such that $\theta_i(e_i) = e_i^*$. By Proposition 3.1, one has $\theta_i(e_j) = 0$

for $j \neq i$, and of course $\theta_i(e_j^*) = 0$ for any j . Put $e_I^{m_I} = \bigotimes_{i \in I} e_i^{\otimes m_I}$ and $e_{I^c}^{*m_{I^c}} = \bigotimes_{i \in I^c} e_i^{*m_i}$. Then one has $e = dz_I \otimes e_I^{m_I} \otimes e_{I^c}^{*m_{I^c}}$ up to an invertible holomorphic function, which does not affect the proof. Recall the local formula of $\bar{\theta}$ (see §1 in [29]):

$$\bar{\theta} = \sum_i \bar{\theta}_i d\bar{z}_i,$$

where $\bar{\theta}_i$ is the adjoint of the matrix θ_i with respect to the Hodge metric. By the product rule for the Higgs field with respect to tensor products (see §1 in [29]), one has for $i \in I^c$,

$$\begin{aligned} \bar{\theta}_i(dz_I \otimes e_I^{m_I} \otimes e_{I^c}^{*m_{I^c}}) &= dz_I \otimes \bar{\theta}_i(e_I^{m_I}) \otimes e_{I^c}^{*m_{I^c}} + dz_I \otimes e_I^{m_I} \otimes \bar{\theta}_i(e_{I^c}^{*m_{I^c}}) \\ &= dz_I \otimes e_I^{m_I} \otimes \bar{\theta}_i(e_{I^c}^{*m_{I^c}}) \\ &= m_i \cdot dz_I \otimes e_I^{m_I} \otimes e_{I^c \setminus \{i\}}^{*m_{I^c} \setminus \{i\}} \otimes (e_i^{*m_i-1} \otimes e_i). \end{aligned}$$

Similarly one gets $\bar{\theta}_i(dz_I \otimes e_I^{m_I} \otimes e_{I^c}^{*m_{I^c}}) = 0$ for $i \in I$. Thus one has the formula

$$\bar{\theta}(e) = \sum_{i \in I^c, m_i \neq 0} m_i (dz_I \wedge d\bar{z}_i \otimes e_I^{m_I} \otimes e_{I^c \setminus \{i\}}^{*m_{I^c} \setminus \{i\}} \otimes e_i^{*m_i-1} \otimes e_i).$$

By the assumption of the case, the above expression is non-zero. The claim is proved. \square

In summary, the space $\mathfrak{h}_{(2)}(I, \emptyset; \mathbb{V}_m)$ is zero unless I is the whole set. This proves the $q = 0$ case. Now we assume $q > 0$. There are two possibilities, namely the case $I \cap J \neq \emptyset$ and the case $I \cap J = \emptyset$. Consider the former case. Let Λ be the adjoint of the Lefschetz operator $L = \wedge \omega$ on the space of differential forms, where ω is the Kähler form of the metric g on X . By the standard L^2 -harmonic theory, $\Lambda(\alpha)$ is again an L^2 -harmonic form. As $I \cap J \neq \emptyset$, $\Lambda(\alpha) = 0$ if and only if $\alpha = 0$. Write $\Lambda(\alpha) = \sum_{I', J'} \beta_{I', J'}$. Then $|J'| = q - 1$ and $\{I', J'\}$ is not a permutation of $\{1, \dots, n\}$. So one proves by induction that each term $\beta_{I', J'}$ of $\Lambda(\alpha)$ is zero and hence $\alpha = 0$. Consider the latter case. Let \mathbb{H}_J^n be the complex manifold whose underlying riemannian structure is the same as that of \mathbb{H}^n but the complex structure differs from the usual one by that at the j -th factor for $j \in J$, one takes the complex conjugate complex structure of \mathbb{H} (see §4 in [18]). One puts $X_{\bar{J}} = \Gamma \backslash \mathbb{H}_J^n$. As observed in §4 [18], such an operation identifies the space of L^2 -harmonic forms $\mathfrak{h}_D(X, \mathbb{V}_m)$ with $\mathfrak{h}_D(X_{\bar{J}}, \mathbb{V}_m)$, but maps the subspace of type $\{I, J; \mathbb{V}_m\}$ on X to the subspace of type $\{I \cup J, \emptyset; \mathbb{V}_m\}$ on $X_{\bar{J}}$. This allows us to reduce the proof to the case $q = 0$ on $X_{\bar{J}}$. However the above arguments, particularly the truth of Proposition 3.6, holds for $X_{\bar{J}}$ as well. Therefore the second case also follows. \square

Now we can proceed to the proof of Theorem 4.3.

Proof. Let $\alpha \in \mathfrak{h}_{(2), \bar{\partial}}^i(X, \mathcal{C}_I)$ be the L^2 -harmonic representative of a non-zero cohomology class of $H_{(2)}^i(X, \mathcal{C}_I^0)$. By Lemma 4.5 $\alpha \in \mathfrak{h}_D^{p, q}(X, \mathbb{V}_m)$ where we rewrite $p = |I|$, $q = i$. By Lemma 4.6, one writes $\alpha = \sum_{I, J, |I|=p, |J|=q} \alpha_{I, J}$ into sum of L^2 -harmonic forms with $\alpha_{I, J}$ type $\{I, J; \mathbb{V}_m\}$.

By Proposition 4.7, $\alpha_{I,\bar{J}} = 0$ unless J is the complement of I . This implies that if $q \neq n - p$, namely $i \neq n - |I|$, $\alpha = 0$ and thus the vanishing part of the theorem. Moreover if $i = n - |I|$, then $\alpha = \alpha_{I,\bar{I}^c}$. Using again the trick of taking the conjugate complex structures on \mathbb{H}^n at the factors I^c (see the proof of Proposition 4.7), one obtains an identification of the space of harmonic forms of type $\{I, I^c; \mathbb{V}_m\}$ on X with that of type $\{(1, \dots, n), \emptyset; \mathbb{V}_m\}$ on $X_{\bar{J}}$. \square

Now we deduce the main results of this section from Theorem 4.3.

Corollary 4.9. *For $k \neq n$, $IH^k(X^*, \mathbb{V}_m) = 0$.*

Proof. By Zucker (5.22) in [32], one has the equality

$$\dim IH^k(X^*, \mathbb{V}_m) = \sum_{P=0}^{k+|m|} \sum_{l=0}^n \sum_{\substack{I \subset \{1, \dots, n\}, \\ |I|=l, |m_I|+|I|=P}} \dim H_{(2)}^{k-l}(X, \mathcal{C}_I^0).$$

In the above formula, it is clear that for $0 \leq k \leq n-1$ fixed and for all l , $k-l < n-l = n-|I|$ holds. By Theorem 4.3, it follows that for $0 \leq k \leq n-1$, each direct summand in the right hand side of the formula is zero, and hence $IH^k(X^*, \mathbb{V}_m) = 0$. By Poincaré duality for intersection cohomology, $IH^k(X^*, \mathbb{V}_m)$ vanishes for $n+1 \leq k \leq 2n$ as well. \square

Proposition 4.10. *One has a natural isomorphism*

$$H_{(2)}^0(X, \mathcal{C}_{\{1, \dots, n\}}^0) \simeq H^0(\bar{X}, \mathcal{O}_{\bar{X}}(-S) \otimes \bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2}).$$

Proof. Let $\iota : X \rightarrow \bar{X}$ be the inclusion. One has the relation

$$H_{(2)}^0(X, \mathcal{C}_{\{1, \dots, n\}}^0) \subset A_{(2)}^{0,0}(X, \mathcal{C}_{\{1, \dots, n\}}^0) = A^0(\bar{X}, \iota_* \mathcal{C}_{\{1, \dots, n\}}^0).$$

We define $\Omega_{(2)}^0(\mathcal{C}_{\{1, \dots, n\}}^0)$ to be the subsheaf of $\iota_* \mathcal{C}_{\{1, \dots, n\}}^0$ consisting of germs of L^2 -holomorphic sections. It is clear that one has a natural isomorphism

$$H_{(2)}^0(X, \mathcal{C}_{\{1, \dots, n\}}^0) \simeq H^0(\bar{X}, \Omega_{(2)}^0(\mathcal{C}_{\{1, \dots, n\}}^0)).$$

Recall that $\mathcal{C}_{\{1, \dots, n\}}^0 = \bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2}|_X$. In the following we show that

$$\Omega_{(2)}^0(\bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2}|_X) = \mathcal{O}_{\bar{X}}(-S) \otimes \bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2}.$$

The question is local at infinity, and we shall only show the case where $m = (1, 0, \dots, 0)$, since the proof for general case is completely analogous. It suffices also to consider a small neighborhood U of a maximal singular point $P \in S$. Let $\{u_1, \dots, u_n\}$ be a set of local coordinates of U such that $S \cap U$ is defined by $\prod_{i=1}^n u_i = 0$. We also put $\omega_{pc} = \sum_{i=1}^n \frac{\sqrt{-1}}{\pi} \frac{du_i \wedge d\bar{u}_i}{|u_i|^2 \log |u_i|^2}$,

the Poincaré metric on $U - S \simeq (\Delta^*)^{\times n}$, and $\omega = c_1 \sum_{i=1}^n \frac{dz_i \wedge d\bar{z}_i}{y_i^2}$ (in the following the c_i are certain constants), the invariant metric. Their volume forms are computed respectively by

$$\text{Vol}_{\omega_{pc}} = \frac{c_2 \bigwedge_{i=1}^n (du_i \wedge d\bar{u}_i)}{\prod_{i=1}^n |u_i|^2 |\log |u_i||^2}, \quad \text{Vol}_{\omega} = \frac{c_3 \bigwedge_{i=1}^n (dz_i \wedge d\bar{z}_i)}{\prod_{i=1}^n y_i^2}.$$

By the theory of toroidal resolutions of a cusp singularity (see [1] and [7]), one has the following formula for the change of local coordinates:

$$2\pi\sqrt{-1} \cdot z_i = \sum_{j=1}^n a_{i,j} \log u_j,$$

where $a_{i,j} > 0$ for all i, j (the rational polyhedral decomposition is taken in the positive cone containing totally positive elements of \mathcal{O}_F). Comparing the real parts of the above equality, one obtains

$$y_i = \sum_{j=1}^n -a'_{i,j} \log |u_j|$$

with $a'_{i,j} = \frac{a_{i,j}}{2\pi}$. The estimate of Hodge norms in Theorem 5.21, [3] is taken over the following type of region

$$D_{\epsilon} = \{(u_1, \dots, u_n) \in (\Delta^*)^n \mid \frac{\log |u_1|}{\log |u_2|} > \epsilon, \dots, \frac{\log |u_{n-1}|}{\log |u_n|} > \epsilon, -\log |u_n| > \epsilon\}$$

for some $\epsilon > 0$. For an element $\sigma \in S_n$, the permutation group over n elements, we put D_{σ} to be the region obtained by permutating the indices of $\{u_i\}$ in the definition of D_{ϵ} . So $D_{id} = D_{\epsilon}$, and note that $\{D_{\sigma}\}_{\sigma \in S_n}$ is an open covering of a small neighborhood of P for suitable chosen ϵ . By shrinking U if necessary, it covers U . It is clear that the square-integrability over U is equivalent to that over each D_{σ} . Now let v_i be a local section of \mathcal{L}_i over U , and write $f(u_1, \dots, u_n) v_1^3 \otimes v_2^2 \otimes \dots \otimes v_n^2$ for an element in $\iota_*(\mathcal{L}_1^3 \otimes \mathcal{L}_2^2 \otimes \dots \otimes \mathcal{L}_n^2)|_X(U)$. By Theorem 5.21, [3], $\|v_i\|^2 \sim |\log |u_i||$ over D_{id} . The condition that $f v_1^3 \otimes v_2^2 \otimes \dots \otimes v_n^2$ being L^2 over D_{id} means that

$$\int_{D_{id}} |f| \cdot \|v_1^3 \otimes v_2^2 \otimes \dots \otimes v_n^2\| \text{Vol}_{\omega} < \infty.$$

Since over D_{id} ,

$$\frac{\text{Vol}_{\omega}}{\text{Vol}_{\omega_{pc}}} \sim \frac{\prod_{i=1}^n |\log |u_i||^2}{\prod_{i=1}^n y_i^2} = \frac{\prod_{i=1}^n |\log |u_i||^2}{\prod_{i=1}^n (\sum_{j=1}^n -a'_{i,j} \log |u_j|)^2} \sim \frac{\prod_{i=1}^n |\log |u_i||^2}{|\log |u_1||^{2n}}$$

holds, it follows that

$$\begin{aligned} \int_{D_{id}} |f| \cdot \|v_1^3 \otimes v_2^2 \otimes \dots \otimes v_n^2\| \text{Vol}_{\omega} &\sim \int_{D_{id}} |f| \cdot |\log |u_1||^{2n+1} \text{Vol}_{\omega} \\ &\sim \int_{D_{id}} |f| \cdot |\log |u_1||^3 \cdot \prod_{i=1}^n |\log |u_i||^2 \text{Vol}_{\omega_{pc}} \\ &\sim \int_{D_{id}} \frac{|f| \cdot |\log |u_1||^3}{\prod_{i=1}^n |u_i|^2} \bigwedge_{i=1}^n (du_i \wedge d\bar{u}_i). \end{aligned}$$

It is clear now that over D_{id} , the above section is L^2 if and only if $f = u_1 \cdot f'$ for certain holomorphic f' . Running the above arguments for the other regions D_σ , one knows that the section is L^2 over all D_σ if and only if $f = (u_1, \dots, u_n) \cdot f''$ for certain holomorphic f'' . This shows the equality

$$\Omega_{(2)}^0(\bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2}|_X) = \mathcal{O}_{\bar{X}}(-S) \otimes \bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2}$$

over U . By the previous discussion, the above equality actually holds over \bar{X} , which shows the proposition. \square

Corollary 4.11. *The Hodge decomposition of $IH^n(X^*, \mathbb{V}_m)$ reads*

$$IH^n(X^*, \mathbb{V}_m) = \bigoplus_{P+Q=n+|m|} IH^{P,Q},$$

where

$$IH^{P,Q} \simeq \bigoplus_{l=0}^n \bigoplus_{\substack{I \subset \{1, \dots, n\}, \\ |I|=l, |m_I|+|I|=P}} H_{(2)}^{n-l}(X, \mathcal{C}_I^0),$$

and has dimension $N(m, P) \dim H^0(\bar{X}, \mathcal{O}_{\bar{X}}(-S) \otimes \bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2})$, where $N(m, P)$ is the cardinality of the set $\{I \subset \{1, \dots, n\} \mid |m_I| + |I| = P\}$. In particular,

$$IH^{n+|m|,0} \simeq H^0(\bar{X}, \mathcal{O}_{\bar{X}}(-S) \otimes \bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2}), \quad IH^{0,n+|m|} \simeq H^n(\bar{X}, \bigotimes_{i=1}^n \mathcal{L}_i^{-m_i}).$$

Finally,

$$\dim IH^n(X^*, \mathbb{V}_m) = 2^n \dim H^0(\bar{X}, \mathcal{O}_{\bar{X}}(-S) \otimes \bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2}).$$

Proof. Continuing the arguments in the above proof, one sees that for each P ,

$$IH^{P,Q} \simeq \bigoplus_{l=0}^n \bigoplus_{\substack{I \subset \{1, \dots, n\}, \\ |I|=l, |m_I|+|I|=P}} H_{(2)}^{n-l}(X, \mathcal{C}_I^0).$$

It is clear that, for $P = 0$ (resp. $P = n + |m|$), the above expression consists of the unique term $H_{(2)}^n(X, \mathcal{C}_\emptyset^0)$ (resp. $H_{(2)}^0(X, \mathcal{C}_{\{1, \dots, n\}}^0)$). By Proposition 4.10, $IH^{n+|m|,0} \simeq H^0(\bar{X}, \mathcal{O}_{\bar{X}}(-S) \otimes \bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2})$. By Serre duality, one has a natural isomorphism $IH^{0,n+|m|} \simeq H^n(\bar{X}, \bigotimes_{i=1}^n \mathcal{L}_i^{-m_i})$. Moreover, for each subset $I \subset \{1, \dots, n\}$, the cohomology group $H_{(2)}^{n-|I|}(X, \mathcal{C}_I^0)$ appears exactly once in the above formula for $P = |m_I| + |I|$. By Theorem 4.3, each direct summand in the right hand side of the formula has the same dimension. It follows that

$$\dim IH^n(X^*, \mathbb{V}_m) = 2^n \dim H^0(\bar{X}, \mathcal{O}_{\bar{X}}(-S) \otimes \bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2}).$$

\square

Corollary 4.12. *One has the dimension formula*

$$\dim H_{(2)}^0(X, \mathcal{C}_{\{1, \dots, n\}}^0) = [h^{n,0}(\bar{X}) + (-1)^n] \prod_{i=1}^n (m_i + 1).$$

Proof. By Corollary 4.9 and 4.11, one has

$$\dim H_{(2)}^0(X, \mathcal{C}_{\{1, \dots, n\}}^0) = (-1/2)^n \sum_{i=1}^{2n} (-1)^i \dim IH^i(X^*, \mathbb{V}_m).$$

For the Euler characteristic, one has the equality

$$\sum_{i=1}^{2n} (-1)^i \dim IH^i(X^*, \mathbb{V}_m) = \text{rank}(\mathbb{V}_m) \sum_{i=1}^{2n} (-1)^i \dim IH^i(X^*, \mathbb{C}).$$

By Theorem 4.1, the computations of the L^2 -cohomology for constant coefficients in §5 and §6, Ch. III of [8] (see particularly Theorem 6.3, Ch. III and Corollary 4.7, Ch. I in loc. cit.), one knows that

$$\dim IH^i(X^*, \mathbb{C}) = \begin{cases} \binom{n}{i/2}, & \text{if } i \neq n \text{ is even;} \\ (-2)^n (\chi(\bar{X}, \mathcal{O}_{\bar{X}}) - 1), & \text{if } i = n \text{ is odd;} \\ (-2)^n (\chi(\bar{X}, \mathcal{O}_{\bar{X}}) - 1) + \binom{n}{n/2}, & \text{if } i = n \text{ is even.} \end{cases}$$

So one obtains

$$\dim H_{(2)}^0(X, \mathcal{C}_{\{1, \dots, n\}}^0) = (-1)^n \chi(\bar{X}, \mathcal{O}_{\bar{X}}) \prod_{i=1}^n (m_i + 1).$$

Finally, by Proposition 4.7, Ch. II in loc. cit., $h^{p,0}(\bar{X}) = 0$ for $1 \leq p \leq n-1$. Hence $\chi(\bar{X}, \mathcal{O}_{\bar{X}}) = 1 + (-1)^n h^{n,0}(\bar{X})$. This proves the formula. \square

5. EISENSTEIN COHOMOLOGY OF THE HILBERT MODULAR GROUP

Let X^\sharp be the Borel-Serre compactification of X with the boundary ∂X^\sharp . Recall that X is homotopy equivalent to X^\sharp and hence one has the natural restriction map $r : H^*(X, \mathbb{V}_{\mathbb{R}}) \rightarrow H^*(\partial X^\sharp, \mathbb{V}_{\mathbb{R}})$. The theory of Eisenstein series (see [10] and [26]) provides the following space decomposition

$$H^*(X, \mathbb{V}_{\mathbb{R}}) = H_!^*(X, \mathbb{V}_{\mathbb{R}}) \oplus H_{\text{Eis}}^*(X, \mathbb{V}_{\mathbb{R}}),$$

where $H_!^*(X, \mathbb{V}_{\mathbb{R}})$ is the image of the cohomology with compact support, and $H_{\text{Eis}}^*(X, \mathbb{V}_{\mathbb{R}})$ maps isomorphically to the image of r . Its elements can be represented using Eisenstein series. In this section, we study the Eisenstein cohomology $H_{\text{Eis}}^*(X, \mathbb{V}_{\mathbb{R}})$. Before doing anything, we first recall the following result, which is a special case of the main theorems in [16]:

Theorem 5.1 (Li-Schwermer [16]). *If \mathbb{V}_m is regular, then $H^i(X, \mathbb{V}_m) = 0$ for $0 \leq i \leq n-1$, and $H^i(X, \mathbb{V}_m) = H_{\text{Eis}}^i(X, \mathbb{V}_m) \stackrel{r}{\simeq} H^i(\partial X^\sharp, \mathbb{V}_m)$ for $i \geq n+1$.*

The following lemma is known by (6.13-18) in [33]:

Lemma 5.2. $H^i(\partial X^\sharp, \mathbb{V}_m) = 0$ unless $m_1 = \cdots = m_n$. As a consequence, $H_{\text{Eis}}^i(X, \mathbb{V}_m) = 0$ if the relation $m_1 = \cdots = m_n$ is not satisfied.

The main result of this section is the following

Theorem 5.3. Assume $m_1 = \cdots = m_n$ and $l \geq n$. Then the restriction map $r : H^l(X, \mathbb{V}_\mathbb{R}) \rightarrow H^l(\partial X^\sharp, \mathbb{V}_\mathbb{R})$ is surjective and $\dim H_{\text{Eis}}^l(X, \mathbb{V}_m) = \binom{n-1}{l-n} h$. Moreover,

$$H_{\text{Eis}}^l(X, \mathbb{V}_m) \subset F^{|m|+n} H^l(X, \mathbb{V}_m)$$

holds, where F^\cdot is the Hodge filtration on $H^l(X, \mathbb{V}_m)$.

After the paper was posted, Wildeshaus informed us that the main result in [2] is also able to show the statement about the Hodge type of the Eisenstein cohomology in the above theorem. The current argument in the proof is based on the treatment of the Eisenstein cohomology for constant coefficients in [8] (see §3 and §4, Ch. III in loc. cit.). In the following we assume $m_1 = \cdots = m_n$. It is clear that the proof of the theorem can be reduced to the statement for the standard cusp ∞ , which is the Γ -equivalence class of (∞, \dots, ∞) . From now on we pretend that ∂X^\sharp has only one component. Let $\Gamma_\infty \subset \Gamma$ be the stabilizer of ∞ and put $X_\infty = \Gamma_\infty \backslash \mathbb{H}^n$.

Proof. We divide the whole proof into several steps.

Step 1: A basis of $H^l(X_\infty, \mathbb{V}_m)$ for $n \leq l \leq 2n - 1$. Let $X_\infty(1)$ be the quotient of the following set

$$\{(z_1, \dots, z_n) \in \mathbb{H}^n \mid \prod_{i=1}^n y_i = 1\}$$

by Γ_∞ , which is naturally identified with ∂X^\sharp . The group Γ_∞ is of the form

$$1 \rightarrow U \rightarrow \Gamma_\infty \rightarrow M \rightarrow 1,$$

where

$$U = \left\{ \begin{pmatrix} 1 & u^{(1)} \\ 0 & 1 \end{pmatrix} \times \cdots \times \begin{pmatrix} 1 & u^{(n)} \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty \mid u \in \mathcal{O}_F \right\}$$

$$M = \left\{ \begin{pmatrix} t^{(1)} & 0 \\ 0 & (t^{-1})^{(1)} \end{pmatrix} \times \cdots \times \begin{pmatrix} t^{(1)} & 0 \\ 0 & (t^{-1})^{(n)} \end{pmatrix} \in \Gamma_\infty \mid t \in \mathcal{O}_F^* \right\}$$

are free abelian groups of rank n and $n - 1$ respectively. The $X_\infty(1)$ has two distinguished submanifolds: One is the quotient of the set $\{(iy_1, \dots, iy_n) \in \mathbb{H}^n \mid \prod_{i=1}^n y_i = 1\}$ by M , which is isomorphic to $(S^1)^{n-1}$ with 'coordinates' $\{\log y_1, \dots, \log y_{n-1}\}$ and denoted temporarily by Y , and the quotient of $\{(x_1 + i, \dots, x_n + i) \in \mathbb{H}^n \mid x_1, \dots, x_n \in \mathbb{R}\}$ by U , which is isomorphic

to $(S^1)^n$ with 'coordinates' $\{x_1, \dots, x_n\}$ and denoted temporarily by Z . In fact, $X_\infty(1)$ is naturally a fiber bundle with Y (resp. Z) a section (resp. fiber) of it (see §2 in [10]).

Claim 5.4. For $n \leq l \leq 2n - 1$, the following set of vector valued differential forms over \mathbb{H}^n is Γ_∞ -invariant and defines a basis of $H^l(X_\infty(1), \mathbb{V}_m)$:

$$\{\omega'_a = \frac{dy_a}{y_a} \wedge dx_1 \wedge \dots \wedge dx_n \otimes \bigotimes_{i=1}^n (e_{i1} + x_i e_{i2})^{m_1} \mid a \subset \{1, \dots, n-1\}, |a| = l - n\},$$

where for $a = (i_1, \dots, i_{l-n})$, $i_1 < \dots < i_{l-n}$, $\frac{dy_a}{y_a} = \frac{dy_{i_1}}{y_{i_1}} \wedge \dots \wedge \frac{dy_{i_{l-n}}}{y_{i_{l-n}}}$, and $\left\{ e_{i1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_{i2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is the standard basis of V_i at $z_0 \in \mathbb{H}^n$ (see §2).

Proof. Let $\pi : X_\infty(1) \rightarrow Y$ be the fibration. By §2 in [10], the Leray spectral sequence of π for $H^l(X_\infty(1), \mathbb{V}_m)$ degenerates at E_2 . By the theorem of van Est (see §2 in [10]), each grading $H^*(Y, R^{l-*}\pi_*\mathbb{V}_m)$ is computed by its corresponding Lie algebra cohomology. By the computations on the Lie algebra cohomology in §6 of [33] (see particularly (6.18) and Lemma (6.13)), one knows that

$$H^l(X_\infty(1), \mathbb{V}_m) = H^{l-n}(Y, \mathbb{C}) \otimes H^n(Z, \mathbb{V}_m).$$

Now it is straightforward to check that $\{\frac{dy_a}{y_a} \mid a \subset \{1, \dots, n-1\}, |a| = l - n\}$ provides a basis for $H^{l-n}(Y, \mathbb{C})$ and the element $\bigwedge_{i=1}^n dx_i \otimes \bigotimes_{i=1}^n (e_{i1} + x_i e_{i2})^{m_1}$ is a basis for the one dimensional space $H^n(Z, \mathbb{V}_m)$. \square

Note that the inclusion $X_\infty(1) \subset X_\infty$ is a homotopy equivalence. We claim the following

Claim 5.5. The following set of Γ_∞ -invariant vector valued differential forms over \mathbb{H}^n

$$\{\omega_a = \frac{dz_a \wedge d\bar{z}_a}{y_a} \wedge dz_b \otimes \bigotimes_{i=1}^n (e_{i1} + z_i e_{i2})^{m_1} \mid a \subset \{1, \dots, n-1\}, |a| = l - n, b = a^c\}$$

defines a basis of $H^l(X_\infty, \mathbb{V}_m)$.

Proof. By Remark 3.1, Ch.III in [8], ω_a is cohomologous to

$$\frac{dy_a}{y_a} \wedge dx_1 \wedge \dots \wedge dx_n \otimes \bigotimes_{i=1}^n (e_{i1} + z_i e_{i2})^{m_1}$$

up to scalar. By Claim 5.4 and the homotopy equivalence, it remains to show that

$$\frac{dy_a}{y_a} \wedge dx_1 \wedge \dots \wedge dx_n \otimes \bigotimes_{i=1}^n (e_{i1} + z_i e_{i2})^{m_1}$$

is cohomologous to ω'_a up to a scalar. Note that the difference of the above two forms is a linear combination of forms of the following type:

$$\frac{dy_a}{y_a} \wedge dx_1 \wedge \cdots \wedge dx_n \otimes \bigotimes_{i=1}^n (y_i e_{i2})^{t_i} \otimes (e_{i1} + x_i e_{i2})^{m_1 - t_i},$$

which is exact once one of the t_i is positive. In fact, assume $t_1 \geq 1$ for example, the exterior differential of the following form

$$y_1 \cdot \frac{dy_a}{y_a} \wedge \bigwedge_{i=2}^n dx_i \otimes (y_1 e_{12})^{t_1 - 1} \otimes (e_{11} + x_1 e_{12})^{m_1 - t_1 + 1} \otimes \bigotimes_{i=2}^n (y_i e_{i2})^{t_i} \otimes (e_{i1} + x_i e_{i2})^{m_1 - t_i}$$

is up to a scalar equal to

$$\frac{dy_a}{y_a} \wedge dx_1 \wedge \cdots \wedge dx_n \otimes \bigotimes_{i=1}^n (y_i e_{i2})^{t_i} \otimes (e_{i1} + x_i e_{i2})^{m_1 - t_i}.$$

This shows the claim. \square

Step 2: Convergence of Eisenstein series. For each ω_a , we consider the following formal differential form $E(\omega_a)$ on X obtained by symmetrization (see §3, Ch.III in [8]):

$$E(\omega_a) = \sum_{M \in \Gamma/\Gamma_\infty} \omega_a|_M,$$

where $M = \begin{pmatrix} a^{(1)} & b^{(1)} \\ c^{(1)} & d^{(1)} \end{pmatrix} \times \cdots \times \begin{pmatrix} a^{(n)} & b^{(n)} \\ c^{(n)} & d^{(n)} \end{pmatrix}$ runs through a set of representatives of Γ/Γ_∞ , and $\omega_a|_M = M^* \omega_a$ by considering M as transformation on X_∞ . If the above series converges, then $E(\omega_a)$ defines a genuine vector valued differential form on X . The following simple transformation formulas

$$\begin{aligned} dz_i|_M &= (c^{(i)} z_i + d^{(i)})^{-2} dz_i, & d\bar{z}_i|_M &= (c^{(i)} \bar{z}_i + d^{(i)})^{-2} d\bar{z}_i, \\ y_i|_M &= |c^{(i)} z_i + d^{(i)}|^{-2} y_i, & (e_{i1} + z_i e_{i2})|_M &= (c^{(i)} z_i + d^{(i)})^{-1} (e_{i1} + z_i e_{i2}) \end{aligned}$$

show that the series $E(\omega_a)$ obeys the relation $E(\omega_a) = E_{\alpha,\beta}(z) \cdot \omega_a$, where

$$E_{\alpha,\beta}(z) = \sum_{M \in \Gamma/\Gamma_\infty} \prod_{i=1}^n (c^{(i)} z_i + d^{(i)})^{-\alpha_i} (c^{(i)} \bar{z}_i + d^{(i)})^{-\beta_i}$$

with

$$\alpha_i = \begin{cases} m_1 + 1 & \text{if } i \in a \\ m_1 + 2 & \text{if } i \in a^c \end{cases}, \quad \beta_i = \begin{cases} 1 & \text{if } i \in a \\ 0 & \text{if } i \in a^c \end{cases}.$$

This is the type of Eisenstein series considered in [8]. Note that for the constant coefficient they consider the border case $r = 1$, which requires the technique of Hecke summation to show the convergence of the series. In the current case, Lemma 5.7, Ch.I in [8] shows that $E_{\alpha,\beta}(z)$ is absolutely convergent. We show next that $E(\omega_a)$ is closed. For that, one considers the

Fourier expansion of $E_{\alpha,\beta}(z)$ at ∞ , and as argued in Proposition 3.3, Ch.III in [8], the key point is to study the constant Fourier coefficient. The formula $a_0(y, s)$ in Page 170, [8] for $s = 0$, $r = \frac{m_1+2}{2}$ and α_i, β_i as above shows that the constant Fourier coefficient of $E_{\alpha,\beta}(z)$ is of the form $A + \frac{B}{\prod_{i=1}^n y_i^{2r-1}}$ with $A = 1$, $B = 0$ (see Theorem 4.9, Ch.III in [8]). One concludes from the proof of Proposition 3.3, Ch.III in [8] that $E(\omega_a)$ is closed (hence a cohomology class in $H^l(X, \mathbb{V}_m)$), and it induces the same cohomology class as ω_a in $H^l(X_\infty, \mathbb{V}_m)$. The same proposition also shows that the restriction of $E(\omega_a)$ to other cusps is zero. By the theory of Eisenstein cohomology, $\{E(\omega_a)\}_{a \in \{1, \dots, n\}, |a|=l-n}$ forms a basis of $H_{\text{Eis}}^l(X, \mathbb{V}_m)$.

Step 3: Hodge type of Eisenstein cohomology classes. By the expression of ω_a in Claim 5.5, one knows that ω_a extends naturally to an element in $A_X^{n,0}(\log S) \wedge \overline{A_X^{|a|,0}(*S)} \otimes E_m^{|m|,0}$. Now by the expression of the Eisenstein series $E_{\alpha,\beta}(z)$ in Theorem 4.9, [8] for $A = 1$, $B = 0$, one sees that $E(\omega_a)$ lies again in $A_X^{n,0}(\log S) \wedge \overline{A_X^{|a|,0}(*S)} \otimes E_m^{|m|,0}$. The expression of $E(\omega_a)$ shows that it is of logarithmic singularity at infinity S . Therefore $E(\omega_a)$ represents a cohomology class in $F^{|m|+n}H^l(X, \mathbb{V}_m)$. \square

6. THE MIXED HODGE STRUCTURE AND THE DIMENSION FORMULAE

By theorem 5.4 in [13] the natural map $IH^k(X^*, \mathbb{V}_m) \rightarrow H^k(X, \mathbb{V}_m)$, induced by the inclusion $j : X \rightarrow X^*$, is a morphism of mixed Hodge structures and the image of the map is the lowest weight piece, that is $W_{k+|m|}$ of $H^k(X, \mathbb{V}_m)$. By Corollary 4.9, the map is only interesting for $k = n$. In this case, we assert the following

Proposition 6.1. *The natural map $IH^n(X^*, \mathbb{V}_m) \rightarrow H^n(X, \mathbb{V}_m)$ is injective.*

Proof. Recall that one has the following natural isomorphism by L^2 -harmonic theory:

$$IH^n(X^*, \mathbb{V}_m) \simeq \mathfrak{h}_{(2)}^n(X, \mathbb{V}_m) = \bigoplus_{p+q=n} \mathfrak{h}_{(2)}^{p,q}(X, \mathbb{V}_m).$$

The proposition boils down to show the following statement: Assume we have $\omega \in \mathfrak{h}_{(2)}^n(X, \mathbb{V}_m)$ and $\alpha \in A_X^{n-1}(\mathbb{V}_m)$ satisfying $D(\alpha) = \omega$, then $\omega = 0$. In order to prove this we write $\omega = \sum_{p+q=n} \omega_{p,q}$ and further $\omega_{p,q} = \sum_{I \subset \{1, \dots, n\}, |I|=p} \omega_{I, \bar{I}^c}$, where $\omega_{I, \bar{I}^c} = f_{I, \bar{J}} dz_I \wedge \overline{dz_{\bar{I}^c}}$ (see Lemma 4.6 and Proposition 4.7). It is enough to show $\omega_{I, \bar{I}^c} = 0$ for all possible I . Let $X_{\bar{I}^c}$ be the complex manifold considered in the proof of Proposition 4.7 for $J = I^c$ and put $\tilde{X} = X_{\bar{I}^c}$ for brevity. Let $\iota : X \rightarrow \tilde{X}$ be the natural inclusion. By Deligne [4], the inclusion

$$\left(\left[\bigoplus_{p+q=n} \mathcal{A}_X^{p,0}(\log S) \wedge \overline{\mathcal{A}_X^{q,0}(*S)} \right] \otimes \bar{\mathbb{V}}_m, D \right) \hookrightarrow (\iota_* \mathcal{A}_X(\mathbb{V}_m), D)$$

is a quasi-isomorphism. Furthermore, by E_1 -degeneration of the Hodge filtration, one has also the quasi-isomorphism

$$\left(\left[\bigoplus_{p+q=\cdot} \mathcal{A}_{\bar{X}}^{p,0}(\log S) \wedge \overline{\mathcal{A}_{\bar{X}}^{q,0}(*S)} \right] \otimes \bar{V}_m, D \right) \simeq \left(\left[\bigoplus_{p+q=\cdot} \mathcal{A}_{\bar{X}}^{p,0}(\log S) \wedge \overline{\mathcal{A}_{\bar{X}}^{q,0}(*S)} \right] \otimes \bar{V}_m, D''_{\bar{X}} \right).$$

It is not difficult to check that $\omega \in [\bigoplus_{p+q=n} \mathcal{A}_{\bar{X}}^{p,0}(\log S) \wedge \overline{\mathcal{A}_{\bar{X}}^{q,0}(*S)}] \otimes \bar{V}_m$. So by the quasi-isomorphisms, we find actually $\alpha' \in [\bigoplus_{p+q=n-1} \mathcal{A}_{\bar{X}}^{p,0}(\log S) \wedge \overline{\mathcal{A}_{\bar{X}}^{q,0}(*S)}] \otimes \bar{V}_m$ such that $D''\alpha' = \omega$. Note for fixed I , ω_{I,\bar{J}^c} is holomorphic over \bar{X} . One has then

$$\begin{aligned} \langle \omega_{I,\bar{J}}, \omega_{I,\bar{J}} \rangle &= \langle (D''\alpha')_{I,\bar{J}}, \omega_{I,\bar{J}} \rangle \\ &= \langle (\bar{\partial}_{\bar{X}}\alpha')_{I,\bar{J}}, \omega_{I,\bar{J}} \rangle + \langle (\theta_{\bar{X}}\alpha')_{I,\bar{J}}, \omega_{I,\bar{J}} \rangle \\ &= \langle \bar{\partial}_{\bar{X}}\alpha', \omega_{I,\bar{J}} \rangle + \langle \theta_{\bar{X}}\alpha', \omega_{I,\bar{J}} \rangle \\ &= \langle \theta_{\bar{X}}\alpha', \omega_{I,\bar{J}} \rangle \\ &= \langle \alpha', \theta_{\bar{X}}^* \omega_{I,\bar{J}} \rangle \\ &= \langle \alpha', 0 \rangle \\ &= 0, \end{aligned}$$

and therefore we get $\omega_{I,\bar{J}} = 0$. So $\omega = 0$, and the proof is completed. \square

Lemma 6.2. *Let $(H_{\mathbb{R}}, W_{\cdot}, F_{\cdot})$ be a MHS with weights $\geq m+k$ and the following properties:*

$$H_{\mathbb{C}} = F^0 = \dots = F^{m+n} \supsetneq F^{m+n+1} = 0, \quad 0 = W_{m+k} \subset \dots \subset W_{2(m+k)} = H_{\mathbb{R}}$$

for certain $\frac{k+1}{2} \leq n \leq k$. Then the weight filtration must be of the form

$$0 = W_{m+k} = \dots = W_{2(m+n)-1} \subsetneq W_{2(m+n)} = \dots = W_{2(m+k)} = H_{\mathbb{R}}.$$

Proof. By the assumption on the Hodge filtration and the Hodge symmetry, it is easy to see that each graded piece of the weight filtration can have at most one Hodge type. This implies that the first possible weight with non-zero dimension is $W_{2(m+n)}$. But then $W_{2(m+n)}$ must be the whole space. This is because for any $i \geq 2(m+n)+1$, the unique Hodge component $(\frac{i}{2}, \frac{i}{2})$ of Gr_i^W (assume i even), which is a quotient of $F^{\frac{i}{2}} \cap W_{i,\mathbb{C}} = 0$, is zero. This implies the result. \square

Proposition 6.3. *Let $(H_{\mathbb{R}}, W_{\cdot}, F_{\cdot})$ be a MHS with weights $\geq m+n$ with*

$$H_{\mathbb{C}} = F^0 \supset \dots \supset F^{m+n} \supsetneq F^{m+n+1} = 0, \quad 0 \subset W_{m+n} \subset \dots \subset W_{2(m+n)} = H_{\mathbb{R}}.$$

Let $H_{\mathbb{R}} = H_{1,\mathbb{R}} \oplus H_{2,\mathbb{R}}$ be a vector space decomposition. Assume that $H_{2,\mathbb{R}} \subset F^{m+n}$ and $H_{1,\mathbb{R}} \subset W_{m+n}$. Then the weight filtration is of the form $0 \subset H_{1,\mathbb{R}} = W_{m+n} = \cdots = W_{2(m+n)-1} \subsetneq W_{2(m+n)} = H_{\mathbb{R}}$, and the MHS $(H_{\mathbb{R}}, W_{\cdot}, F_{\cdot})$ is split over \mathbb{R} .

Proof. Consider the quotient MHS on $(\frac{H_{\mathbb{R}}}{W_{m+n}}, \tilde{W}_{\cdot}, \tilde{F}_{\cdot})$, where \sim means the quotient filtration. By the assumption on $H_{1,\mathbb{R}}$ and $H_{2,\mathbb{R}}$, one sees that the above quotient MHS is of the form in Lemma 6.2. Thus one obtains the assertion about the weight filtration W_{\cdot} except the equality $W_{m+n} = H_{1,\mathbb{R}}$.

Set $H^{p,q} := F^p \cap \bar{F}^q \cap W_{p+q,\mathbb{C}}$. W_{m+n} has a pure Hodge structure of weight $m+n$ induced by F_{\cdot} and its Hodge (p,q) -component on W_{m+n} is given by $F^p \cap \bar{F}^q \cap W_{m+n,\mathbb{C}}$. So $W_{m+n,\mathbb{C}} = \bigoplus_{p+q=m+n} H^{p,q}$ and $H^{p,q} \cap W_{m+n,\mathbb{C}} = 0$ for $p+q \neq m+n$. Because for $m+n < p+q < 2(m+n)$ $W_{p+q} = W_{m+n}$, one has $H^{p,q} = H^{p,q} \cap W_{m+n,\mathbb{C}} = 0$. Now consider the weight $2(m+n)$ pure Hodge structure on $Gr_{2(m+n)}^W$. By Hodge symmetry and the indexing of the Hodge filtration, $H^{p,2(m+n)-p}$ is zero unless $p = m+n$. In that case, $H^{m+n,m+n} = F^{m+n} \cap \bar{F}^{m+n}$. Note that $H_{2,\mathbb{R}} \subset H^{m+n,m+n}$ by the assumption. This implies the following relation:

$$W_{2(m+n),\mathbb{C}} = H_{\mathbb{C}} = H_{1,\mathbb{C}} \oplus H_{2,\mathbb{C}} \subset W_{m+n,\mathbb{C}} \oplus H^{m+n,m+n} \subset W_{2(m+n),\mathbb{C}}.$$

Therefore $H_{2,\mathbb{R}} = W_{m+n}$ and $H_{2,\mathbb{C}} = H^{m+n,m+n}$ hold, which also shows the relation $W_{l,\mathbb{C}} = \bigoplus_{p+q \leq l} H^{p,q}$ for each $m+n \leq l \leq 2(m+n)$. To show that $(H_{\mathbb{R}}, W_{\cdot}, F_{\cdot})$ is split over \mathbb{R} , it remains to show that $F^p = \bigoplus_{r \geq p} H^{r,s}$ holds for each p (see §2 in [3]). Because $H = W_{m+n,\mathbb{C}} \oplus H_{2,\mathbb{C}}$ as shown above, and $H_{2,\mathbb{C}} \subset F^{m+n}$ by assumption, it follows that $F^p = F^p(W_{m+n,\mathbb{C}}) \oplus H_{2,\mathbb{C}}$ for each p . Now that $F^p(W_{m+n,\mathbb{C}}) = \bigoplus_{r \geq p} H^{r,m+n-r}$, one obtains then $F^p = \bigoplus_{r \geq p} H^{r,s}$ for each p . This proves the result. \square

Now we proceed to deduce our main results of the paper from the above results, together with the established information in previous sections. Let us return to the decomposition (see §5):

$$H^k(X, \mathbb{V}_{\mathbb{R}}) = H_{\dagger}^k(X, \mathbb{V}_{\mathbb{R}}) \oplus H_{\text{Eis}}^k(X, \mathbb{V}_{\mathbb{R}}).$$

By Proposition 6.1, we denote again by $IH^k(X^*, \mathbb{V}_{\mathbb{R}})$ the image of it in $H^k(X, \mathbb{V}_{\mathbb{R}})$. The cohomology classes in $H_{\dagger}^k(X, \mathbb{V}_{\mathbb{R}})$ are representable by differential forms with compact support, which are square integrable with respect to *any* complete Kähler metric on X . Recall that by Theorem 4.1, the classes of $IH^k(X^*, \mathbb{V}_{\mathbb{R}})$ are L^2 -cohomology classes with respect to the invariant metric. Therefore $H_{\dagger}^k(X, \mathbb{V}_{\mathbb{R}}) \subset IH^k(X^*, \mathbb{V}_{\mathbb{R}})$. The following result improves Theorem 5.1:

Theorem 6.4. *For $k \neq n$, one has $H^k(X, \mathbb{V}_m) = H_{\text{Eis}}^k(X, \mathbb{V}_m)$. Furthermore, for $0 \leq k \leq n-1$ and $k = 2n$, it holds $H^k(X, \mathbb{V}_m) = 0$, and for $n+1 \leq k \leq 2n-1$, $H^k(X, \mathbb{V}_m) = H_{\text{Eis}}^k(X, \mathbb{V}_m) \stackrel{r}{\simeq} H^k(\partial X^\sharp, \mathbb{V}_m)$.*

Proof. By the above discussion, one knows that

$$\dim H_{\text{Eis}}^k(X, \mathbb{V}_m) \leq \dim H^k(X, \mathbb{V}_m) \leq \dim IH^k(X^*, \mathbb{V}_m) + \dim H_{\text{Eis}}^k(X, \mathbb{V}_m).$$

By Corollary 4.9, it follows that $H^k(X, \mathbb{V}_m) = H_{\text{Eis}}^k(X, \mathbb{V}_m)$ for $k \neq n$. The remaining part of the theorem follows from Lemma 5.2 and Theorem 5.1. Also one notices that $H_{\text{Eis}}^{2n}(X, \mathbb{V}_m) = 0$, since ∂X^\sharp is of real dimension $2n-1$. \square

Corollary 6.5. *For $l < n$, $H^l(\bar{X}, \bigotimes_{i=1}^n \mathcal{L}_i^{-m_i}) = 0$. Let c_i be the i -th chern class of \bar{X} , c'_i be the i -th chern class of $T_{\bar{X}}(-\log S)$, the dual vector bundle of $\Omega_{\bar{X}}^1(\log S)$, and $P(c_1, \dots, c_n)$ be the degree n polynomial computing $\chi(\bar{X}, \mathcal{O}_{\bar{X}})$ by Hirzebruch-Riemann-Roch. Then one has $\prod_{i=1}^n \mathcal{L}_i = \chi(\bar{X}, \mathcal{O}_{\bar{X}})$ and $P(c_1, \dots, c_n) = P(c'_1, \dots, c'_n)$.*

Proof. By Proposition 2.2 and 3.3, $\dim H^l(\bar{X}, \bigotimes_{i=1}^n \mathcal{L}_i^{-m_i}) \leq \dim H^l(X, \mathbb{V}_m)$, which is zero for $0 \leq l \leq n-1$ by Theorem 6.4. Thus $H^l(\bar{X}, \bigotimes_{i=1}^n \mathcal{L}_i^{-m_i}) = 0$ for $l < n$. By Hirzebruch-Riemann-Roch theorem, one has

$$\begin{aligned} \dim H^0(\bar{X}, \mathcal{O}_{\bar{X}}(-S) \otimes \bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2}) &= \dim H^n(\bar{X}, \bigotimes_{i=1}^n \mathcal{L}_i^{-m_i}) \\ &= (-1)^n \chi(\bar{X}, \bigotimes_{i=1}^n \mathcal{L}_i^{-m_i}) \\ &= (-1)^n \left[\prod_{i=1}^n (m_i + 1) - 1 \right] \prod_{i=1}^n \mathcal{L}_i + (-1)^n \chi(\bar{X}, \mathcal{O}_{\bar{X}}). \end{aligned}$$

By Proposition 4.10 and Corollary 4.12, it follows that

$$\left[\prod_{i=1}^n (m_i + 1) - 1 \right] \prod_{i=1}^n \mathcal{L}_i = \left[\prod_{i=1}^n (m_i + 1) - 1 \right] \chi(\bar{X}, \mathcal{O}_{\bar{X}}),$$

which implies the equality $\prod_{i=1}^n \mathcal{L}_i = \chi(\bar{X}, \mathcal{O}_{\bar{X}})$ for at least one of m_i is positive. By Hirzebruch proportionality in the non-compact case (see Theorem 3.2 in [20]) and Proposition 3.1, it follows that $\prod_{i=1}^n \mathcal{L}_i = P(c'_1, \dots, c'_n)$, which implies the second equality. \square

Theorem 6.6. *Let \mathbb{V}_m be an irreducible non-trivial local system as above and $(H^k(X, \mathbb{V}_m), W_\cdot, F_\cdot)$ be the Deligne-Saito MHS with weights $\geq |m| + k$. Then:*

- (i) *For $n+1 \leq k \leq 2n-1$, one has $H^k(X, \mathbb{V}_{\mathbb{R}}) = H_{\text{Eis}}^k(X, \mathbb{V}_{\mathbb{R}})$ and the MHS on $H^k(X, \mathbb{V}_{\mathbb{R}})$ is pure and of pure type $(|m| + n, |m| + n)$.*

- (ii) $IH^n(X^*, \mathbb{V}_{\mathbb{R}}) = H_!^n(X, \mathbb{V}_{\mathbb{R}})$ and $H^n(X, \mathbb{V}_{\mathbb{R}}) = IH^n(X^*, \mathbb{V}_{\mathbb{R}}) \oplus H_{\text{Eis}}^n(X, \mathbb{V}_{\mathbb{R}})$ is the splitting of the weight filtration over \mathbb{R} .
- (iii) For $n \leq k \leq 2n - 1$ the weight filtration is of the following form:

$$0 \subset W_{|m|+k} = \cdots = W_{2(|m|+n)-1} \subset W_{2(|m|+n)} = \cdots = W_{2(|m|+k)} = H^k(X, \mathbb{V}_m),$$

where $W_{|m|+k} = IH^k(X^*, \mathbb{V}_m)$ and $Gr_{2(|m|+n)}^W \simeq H_{\text{Eis}}^k(X, \mathbb{V}_m)$. The Hodge filtration is of the following form:

$$H^k(X, \mathbb{V}_m) = F^0 \supset \cdots \supset F^{|m|+n} \supset 0,$$

in which $H_{\text{Eis}}^k(X, \mathbb{V}_m) \subset F^{|m|+n}$ holds.

Proof. For $n + 1 \leq k \leq 2n - 1$, (i) and (iii) follows from Theorem 6.4, 5.3 and Lemma 6.2. For $k = n$, one applies Proposition 6.3 for $H_{\mathbb{R}} = H^n(X, \mathbb{V}_{\mathbb{R}})$, $H_{1,\mathbb{R}} = H_!^n(X, \mathbb{V}_{\mathbb{R}})$ and $H_{2,\mathbb{R}} = H_{\text{Eis}}^n(X, \mathbb{V}_{\mathbb{R}})$. The condition for H_2 follows from Theorem 5.3. The relation $H_{1,\mathbb{R}} \subset IH^n(X^*, \mathbb{V}_{\mathbb{R}}) \subset W_{|m|+n}$ follows from the above discussion. Then Proposition 6.3 implies that

$$H_{1,\mathbb{R}} = IH^n(X^*, \mathbb{V}_{\mathbb{R}}) = W_{|m|+n},$$

and the splitting of the MHS over \mathbb{R} . \square

For the MHS $(H^k(X, \mathbb{V}_m), W, F \cdot)$, put

$$h_k^{P,Q} := \dim Gr_F^P Gr_{\bar{F}}^Q Gr_{P+Q}^W H^k(X, \mathbb{V}_m), \quad H_k^{P,Q} := F^P \cap \bar{F}^Q \cap W_{P+Q,\mathbb{C}}.$$

By Theorem 6.6, $\dim H_k^{P,Q} = h_k^{P,Q}$.

Theorem 6.7. *Let \mathbb{V}_m be an irreducible non-trivial local system over X . Then the following statements hold:*

- (i) $H^k(X, \mathbb{V}_m) = 0$ for $0 \leq k \leq n - 1$ and $k = 2n$.
- (ii) If $m_1 = \cdots = m_n$, then for $n + 1 \leq k \leq 2n - 1$

$$h_k^{|m|+n, |m|+n} := \dim_{\mathbb{C}} F^{|m|+n} W_{2(|m|+n)} H^k(X, \mathbb{V}_m) = \dim_{\mathbb{C}} H^k(X, \mathbb{V}_m) = \binom{n-1}{k-n} h,$$

where h is the number of cusps.

- (iii) If not all m_i are equal, then $H^k(X, \mathbb{V}_m) = 0$ for $n + 1 \leq k \leq 2n - 1$.
- (iv) Furthermore, if $m_1 = \cdots = m_n$, then

$$\dim H^n(X, \mathbb{V}_m) = [2(m_1 + 1)]^n [h^{n,0}(\bar{X}) + (-1)^n] + h.$$

Moreover, for each $P = (m_1 + 1)l_P$, $0 \leq l_P \leq n$, and $P + Q = |m| + n$,

$$h_n^{P,Q} = \binom{n}{l_P} (m_1 + 1)^n [h^{n,0}(\bar{X}) + (-1)^n],$$

$$h_n^{|m|+n, |m|+n} = h \text{ and otherwise } h_n^{P,Q} = 0.$$

(v) If $m_1 = \dots = m_n$ is not satisfied, then

$$\dim H^n(X, \mathbb{V}_m) = 2^n [h^{n,0}(\bar{X}) + (-1)^n] \prod_{i=1}^n (m_i + 1).$$

Furthermore, for $P + Q = |m| + n$,

$$h_n^{P,Q} = N(m, P) [h^{n,0}(\bar{X}) + (-1)^n] \prod_{i=1}^n (m_i + 1),$$

where $N(m, P)$ is the cardinality of the set $\{I \subset \{1, \dots, n\} \mid |m_I| + |I| = P\}$, and otherwise $h_n^{P,Q} = 0$, if $P + Q \neq |m| + n$.

Proof. (i), (ii) and (iii) are consequences of Theorem 6.4, 6.6, 5.3 and Lemma 5.2. (iv) and (v) follow from Corollary 4.11, 4.12, Proposition 4.10, Theorem 6.6 (ii), 5.3 and Lemma 5.2. \square

Theorem 6.8. *One has the following natural isomorphisms:*

- (i) For $n + 1 \leq k \leq 2n - 1$, $H_k^{|m|+n, |m|+n} \simeq H^{k-n}(\bar{X}, \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2})$.
- (ii) $H_n^{|m|+n, 0} \simeq H^0(\bar{X}, \mathcal{O}_{\bar{X}}(-S) \otimes \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2})$, $H_n^{|m|+n, |m|+n} \simeq H^0(S, \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2}|_S)$,
and for $0 \leq P \leq |m| + n - 1$, $P + Q = |m| + n$,

$$H_n^{P,Q} \simeq \bigoplus_{\substack{I \subset \{1, \dots, n\}, \\ |m_I| + |I| = P}} H^{k-|I|}(\bar{X}, \bigotimes_{i \in I} \mathcal{L}_i^{m_i+2} \otimes \bigotimes_{i \in I^c} \mathcal{L}_i^{-m_i}).$$

Proof. (i) follows directly from Theorem 6.6 (i) and Corollary 3.4. By Theorem 6.6, one has for $0 \leq P \leq |m| + n - 1$ and $P + Q = |m| + n$,

$$H_n^{P,Q} = Gr_F^P H^n(X, \mathbb{V}_m).$$

The isomorphisms for these $H_n^{P,Q}$ follow from Corollary 3.4. By Theorem 6.6 and 6.7, one has

$$Gr_F^{|m|+n} H^n(X, \mathbb{V}_m) = F^{|m|+n} = H_n^{|m|+n, 0} \oplus H_n^{|m|+n, |m|+n},$$

and $H_n^{|m|+n, 0} = IH^{|m|+n, 0}(X^*, \mathbb{V}_m)$. By Corollary 4.11, $H_n^{|m|+n, 0} \simeq H^0(\bar{X}, \mathcal{O}_{\bar{X}}(-S) \otimes \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2})$, and by Corollary 3.4, $Gr_F^{|m|+n} H^n(X, \mathbb{V}_m) \simeq H^0(\bar{X}, \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2})$ (the previous two isomorphisms are in fact equalities). As $H^1(\bar{X}, \mathcal{O}_{\bar{X}}(-S) \otimes \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2}) = 0$ by Corollary 6.5, the long exact sequence of sheaf cohomologies of the following short exact sequence

$$0 \rightarrow \mathcal{O}_{\bar{X}}(-S) \otimes \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2} \rightarrow \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2} \rightarrow \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2}|_S \rightarrow 0$$

yields the isomorphism $H_n^{|m|+n, |m|+n} \simeq H^0(S, \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2}|_S)$. \square

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